Stochastic multifractality and universal scaling distributions

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A hierarchical random process can be characterized by a scaling probability distribution describing the stochastic multiplicative process occurring at each generation. Replica averages of the associated random partition functions yield multifractal spectra $\tau(q,n)$ dependent on the number of members n in the replica average chosen, with the exponents generated by quenched $(n=0)$ and annealed $(n=1)$ averaging being but two points in this continuous spectrum. As a consequence the α versus $f(\alpha)$ description has to be generalized for stochastic multifractals. Robust properties of these multifractal spectra, including the position and singularities of generalized dimensions at phase transitions, spectral inequalities and asymptotics, and the scaling behavior of minimal probabilities, are found to depend on universal properties of the scaling probability distribution, specifically on the form of singularities in the distribution, the strength of correlations between measure and length scale during fragmentation, and whether or not the process is conservative. We apply our approach to both the mass and growth scaling probability distributions for diffusion-limited aggregation and show that their construction is Markovian, but while the mass measure is log normal, the singularities in the growth measure do not obey the central limit theorem.

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I. INTRODUCTION

Some of the most intriguing and yet least understood structures that make an appearance in physics far from thermal equilibrium appear to be stochastically self-similar. These structures include diffusion-limited aggregates [1,2], percolating networks [3–5], and the dissipation and passive scalar fields in the inertial range of turbulence [6–10].

Not only are the structures fascinating to look at, but they control a surprising variety of physical phenomena. Thus diffusion-limited aggregation [11] (DLA) appears to be the basic paradigm for such apparently diverse phenomena as two fluid flow in porous media [12], electrochemical deposition [13,14], dielectric breakdown [15], and retinal vasculature [16]. Percolation theory has been invoked to account for epidemics and forest fires [17], kinetic gelation in the sol-gel transition [18], fluid flow in porous media by invasion percolation [19–21], as well as aspects of disordered transport including diffusion on and the conductivity of random resistor networks [22–25], and the elastic properties of random networks [26,27]. Turbulent flows control phenomena as varied as the anomalous diffusion observed at high Reynolds numbers [28,29], the breakup of large scale fluid inhomogeneities during mixing processes [30] including the interesting case of critical binary mixtures [31,32], the structure of clouds [33,34], and the scattering of light and sound waves due to the spatial and temporal fluctuations induced in passive scalars by the turbulent flow [35–37].

The observed self-similarity in structure and physical properties can be understood in each case in terms of the existence of a multifractal measure [8,38–40] such as the growth probability distribution in the case of DLA [41,42], the current fluctuations in random resistor networks [25], or the velocity fluctuations in a turbulent flow field [9,10,43].

This distribution can be found experimentally by covering the structure of outer length scale L with boxes of size l and counting the measure $\mu_i(l)$ in each box i. The scaling properties of the distribution can then be found in the multifractal formalism in terms of the Renyi dimensions [44,38] D_q from the partition function $Z_q = \sum_i \mu_i(l)^q \sim (l/L)^{\tau_q} \sim (l/L)^{(q-1)D_q}$.

Another approach [45,46] to characterizing multifractal measures is in terms of the number of measure singularities $N(\alpha,l) \approx \rho(\alpha)(l/L)^{-f(\alpha)}$ of strength $\mu(l) \sim (l/L)^{\alpha}$. Mathematically the descriptions are simply Legendre transformations of one another, but physically the $f(\alpha)$ versus α description suggests the existence of intertwined sets with fixed singularity α and associated fractal dimension $f(\alpha)$.

The introduction of the thermodynamic formalism for multifractals [47,48] allowed the language and more importantly the ideas of statistical mechanics to be applied to multifractals. For example, equating the partition function in the multifractal formalism with that in statistical mechanics implies analogies [49] between the inverse temperature β and q, the free energy $F(\beta)$ and $\tau(q)$, the energy E and α, and the entropy $S(E)$ and $f(\alpha)$.

The concept of phase transitions can also be transferred from statistical mechanics to describe multifractal behavior [50]: thus singularities in $F(\beta)$ suggest that similar singularities may be found in the $\tau(q)$ spectrum, and the complete scaling spectrum between $-\infty < q < \infty$ may disappear to be replaced, if a multifractal phase transition occurs, by a partial spectrum for $\tau(q)$ existing only for $q > q_{\text{bottom}}$ or $q < q_{\text{top}}$. Indeed, such singularities have been found in both random resistor networks [25] and DLA [51–54].

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An important feature of the work on DLA has been the observed dependence of multifractal phase transitions on the nature of the average taken. Using exact enumeration techniques over an annealed ensemble of all clusters, Lee and Stanley [51] suggested that for $q < q_{\text{bottom}} \approx -1.0$ multifractal scaling breaks down. Trunfio and Alstrøm [53] considered the distribution of growth probabilities in single clusters and proposed that, in this quenched limit, a first-order phase transition existed at $q = 0$. This transition at $q = 0$ was confirmed by careful simulations of quenched partition functions by Schwarzer *et al.* [54]. These results emphasize the importance of the averaging process when studying stochastic multifractals, while the thermodynamic formalism suggests that the methods created specifically to study disorder in spin glasses may prove fruitful here also.

In spin-glass theory stochasticity in the form of random spin-spin interactions J_{ij} in the Hamiltonian $H = \sum_{ij} J_{ij}\sigma_i\sigma_j$ mean that the partition function $Z(\beta, \{J_{ij}\}) = \text{Tr}e^{-\beta H}$ is itself a random variable. Experimental averages of physical variables in the case of interactions quenched over the time scale of the experiment can be found from the configurationally averaged free energy $F_Q(\beta) = (-1/\beta)\langle\langle \ln Z \rangle\rangle$, where $\langle\langle\ \rangle\rangle$ denotes a configurational average over all realizations of the quenched microscopic interactions. In contrast, if the interactions were annealed, then physical variables would be calculated from the configurationally averaged partition function and $F_A(\beta) = (-1/\beta)\ln\langle\langle Z \rangle\rangle$.

Unfortunately the average over the quenched random interactions cannot be performed before the calculation of the partition function is made because of the intervening logarithm. If, as was first noticed by Edwards and Anderson [55], one calculates the free energy $F(\beta, n) = (-1/\beta n)\ln\langle\langle Z^n \rangle\rangle$ of a configurationally averaged integer number n of replicas instead, then these averaging processes can be interchanged. The quenched average can then be calculated provided the analytic continuation $F_Q(\beta) = \lim_{n \to 0} F(\beta, n)$ can be made.

We may use replica averages to study stochastic multifractals also. There is a major difference between the two fields, however. In spin-glass theory, the form of the spin Hamiltonian and the probability distribution controlling the random interactions are assumed to be known. The task is to find the configurationally averaged free energy in the case of nontrivial random interactions. The task faced when studying DLA, percolation, or turbulence is that the nature of the hierarchical dynamical process assumed to be creating the multifractal distribution remains to a large degree unknown. Consequently we are in general more interested in the inverse problem of characterizing the stochastic process creating the multifractal distribution.

The question is how should these stochastic processes be characterized? If it is assumed that these physical processes are both hierarchical and random, then there will exist an associated scaling probability distribution (SPD) which encodes the complete stochastic multiplicative process at each generation by specifying the probability of splitting into a given number of fragments of given measure and length scale. All physical properties of the dynamics including the multifractal spectra can be recreated if we are supplied with this SPD. In the language of algorithmic information theory we have found the minimal algorithm for the process, or at least a very good upper bound on its algorithmic complexity.

But how do we find this SPD? The natural way might be to try and use the information stored in experimentally determined multifractal spectra. The basic problem with this approach is that the D_q versus q and the associated $f(\alpha)$ versus α spectrum measure statistical properties of the underlying multiplicative process, whose shape is fairly insensitive to sometimes quite large changes in this hierarchical process. As a consequence this approach has proved impracticable. Thus it has been shown by Chabra, Jensen, and Sreenivasan [56] in the context of random fragmentation that one cannot even extract unambiguously the mean number of splittings at each level of refinement without additional dynamical data or such information as whether the splitting probabilities are Markovian. In general many different models may be made, in each case a multiparameter fit to the spectrum generated, which in each case can then usually be fitted to within experimental accuracy to the spectrum. A more productive approach, therefore, may be to consider whole classes of multiplicative processes and try to identify universal properties. This approach is followed in this paper and is much more in the spirit of dynamical systems theory where attempts at identifying the specific mapping associated with a given transition to chaos are replaced by the identification of various universal routes to chaos.

For concreteness we shall study a form of random weighted curdling where the measure and the length scale splittings are chosen from a Markovian probability distribution at each level of refinement. This class of models is sufficiently general and rich in structure as to have the capability of characterizing such diverse phenomena as the mass and growth distributions in DLA and perhaps the stochastic vortex fragmentation in turbulence, yet tractable enough that the influence of universal features of an SPD on dynamics can be extracted and studied. Such universal features would include, but not be limited to, questions such as the relationship between multifractal phase transitions and singularities in the SPD. The structure of the paper is therefore as follows.

In Sec. II we introduce the universality classes for the scaling probability distributions we wish to examine and derive the major identities of the theory involving n replica averages. In Sec. III we concentrate on the behavior of the fractal and information dimensions as functions of the replica averages taken. In Sec. IV we show explicitly that the usual $f(\alpha)$ versus α curves often used to characterize multifractals are only valid for deterministic processes and are actually incompatible with stochastic ones. In Sec. V we concentrate on the exponent inequalities and their asymptotic forms as functions of the averaging process employed. In Sec. VI we study an important set of universality classes which involve independent stochastic fragmentation; the structure of the SPD in this case allows an explicit analysis of the dependence of generating functions for multifractal spectra on splitting multiplicities. In Sec. VII we emphasize quenched averaging because of its experimental importance and consider in detail the problem of quenched averages for independent stochastic fragmentation; the analytic continuation $n \rightarrow 0$ involved is accomplished by the use of dispersion relations for the resulting characteristic functions. In Sec. VIII, we study the relationship between multifractal phase transitions, the scaling observed in the minimal measure fragments generated by multiplicative stochastic processes, and the underlying SPD generating the data; these relationships are some of the most useful in extracting the singularities of the underlying SPD. In Sec. IX we consider the consequences of conservation for stochastic multiplicative processes by studying the case of conserved binary fragmentation in detail. In Sec. X, as an explicit example of such conserved processes, we consider the SPDs for the mass and growth probability distribution of DLA; one consequence of the form of these SPDs for DLA is the explicit appearance of multifractal phase transitions in the growth multifractal spectrum, while the much more homogeneous behavior of the mass spectrum can also be accounted for. Finally, in Sec. XI we summarize the results of this paper and offer some concluding remarks.

II. REPLICA AVERAGES AND SCALING DISTRIBUTIONS

The original Cantor construction for creating a fractal dust of dimension $D = \ln 2/\ln 3$ on a line of length l_0 consisted at the first stage of the construction of dividing the line into three and throwing out the middle third. This process was repeated recursively on each subsequent line segment. Thus at the $(N+1)$th stage of construction the remaining 2^N portions of the line of length $l_0 3^{-N}$ were each divided into three and each middle third removed. Besicovitch, instead of throwing out the middle third, introduced deterministic weighted curdling by associating an initial measure μ_{total} with the line l_0, dividing this measure into three fractions f_1, f_2, f_3, and distributing it conservatively $f_1 + f_2 + f_3 = 1$ onto the three line segments. The result after repeating this space filling (in $d = 1$ embedding space) process recursively will be a singular measure whose fractal dimension, in contrast to the Cantor construction, is the same as its embedding dimension $D = 1$, but whose information dimension is $D_I = -[f_1 \ln f_1 + f_2 \ln f_2 + f_3 \ln f_3]/\ln 3$. This procedure was generalized into random weighted curdling by Mandelbrot [8,57] and studied extensively by Kahane and Peyriere [58] by making the multiplicative procedure stochastic in both measure fragmentation and length scale fragmentation. In addition there is no need to associate the process with a one dimensional embedding space such as a line, and consequently random weighted curdling has great flexibility as a descriptor for multiplicative stochastic processes.

Consider an object of initial measure μ_{total} (the measure could represent the total growth probability measure on a DLA cluster, the total instantaneous dissipation in some volume of a turbulent flow, the total current through a percolation cluster, etc.) embedded in a volume of length scale l_0 and dimension d which fragments into a maximum of m smaller pieces in a stochastic multiplicative manner. Each such fragment of the incipient multifractal is reduced in length scale by different random ratio r and contains a different fraction f of the measure at the previous hierarchical level. The word "fragmentation" does not need to be taken literally here. For example, in a DLA cluster because of its treelike topology (see Sec. X) branching will automatically split the total growth probability of a branch into the growth probabilities on the subbranches of which the branch considered is composed (the growth probability on the node itself can be distributed equally among its descent branches) and this is a form of fragmentation. The number of fragments m may physically be some small number such as $m = 2$ for a known binary process as has been suggested is the case for turbulence [59] or $m = 2d - 1$ for DLA on a hypercubic lattice in d dimensions; it is useful, however, not to fix this parameter here but rather treat it as variable whose influence on stochastic multifractality needs to be examined.

After N levels of fragmentation a specific piece of the developing multifractal can be described by a unique string $\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_N$ with $1 \le \alpha_i \le m$. This fragment will have measure

$$\mu_{\boldsymbol{\alpha}} = \mu_{\text{total}} \prod_{i=1}^{N} f_{\alpha_i} \tag{1}$$

and length scale

$$l_{\boldsymbol{\alpha}} = l_0 \prod_{i=1}^{N} r_{\alpha_i} \ . \tag{2}$$

At this point we make the explicit assumption that this multiplicative process is Markovian. This is certainly not the only possibility but, apart from reasons of simplicity, this class of processes exhibits a theoretical structure that hopefully is sufficiently rich to accommodate the physical stochastic multifractals studied to date. Because of the Markovian nature of the splitting, they are fully described by an SPD

$$P(f_1, \ldots, f_m; r_1, \ldots, r_m) \prod_{\alpha} df_\alpha \prod_{\alpha} dr_\alpha \ , \tag{3}$$

which describes the fragmentation process at each generation and can be used to encode the complete stochastic process, and whose singularities, as we shall show, control the observed multifractal phase transitions.

The SPD given by Eq. (3) is defined over a $2m$ dimensional unit hypercube $0 < r_\alpha < 1$, $0 < f_\alpha < 1$, but in general the physical mechanisms controlling fragmentation will constrain the manifold on which the SPD is nonzero and the nature of any singularities on this manifold, thus defining universality classes. Specifically we identify three categories which can be used to classify universality classes: (A) the smoothness or the existence of singularities in $P(f_1,\ldots,f_m;r_1,\ldots,r_m)$; (B) the degree of correlation in the fragmentation between r and f; and (C) conservation laws. These are not unique choices, but do serve to distinguish major types of physical processes; future work may identify additional characteristics which are particularly useful in describing the stochastic multiplicative dynamics of physical processes. Let us discuss them in more detail.

A. Singularities in the distribution

The classification of singularities in SPDs describing physical processes such as DLA or turbulence is at present unknown. Nevertheless, it appears useful to distinguish two major forms for such SPDs by their smoothness and continuity.

(i) The domain in which the SPD is nonzero is bounded both below and above: $P(f_1,\ldots,f_m;r_1,\ldots,r_m)\neq 0$ only for $r_{\min}\leq r_\alpha \leq r_{\max}$ and $f_{\min}\leq f_\alpha \leq f_{\max}$, with finite values of the probability at the domain boundary and no singularities in the domain. Thus at each generation the probability of the measure splitting into infinitesimal fractions and the associated length scale splitting into infinitesimal ratios is expressly forbidden as are all fragments whose dimensions are above a certain ratio of the parents length scale or containing more than a maximal allowed measure fraction. Let us call this a type A process.

(ii) The region in which the SPD is nonzero reaches at least at some edges of the integration domain and may contain singularities there. It is such singularities that generate very strong ($r\to 0$, f finite) and very weak ($f\to 0$, r finite) measure fragments leading to singularities in the multifractal spectrum. We call this a type B process.

B. Correlations in the SPD

Another major feature of SPDs can be expected to be the nature of correlations between the measure fraction f and the length scale ratio r.

(i) The extreme limit of strong correlation would be homogeneous or H stochastic multifractals for which a given length scale is associated with a given measure $f=g(r)$ deterministically reducing the $2m$ dimensional SPD domain to an m dimensional submanifold. In this case

$$P(f_1,\ldots,f_m;r_1,\ldots,r_m)=P(r_1,\ldots,r_m)\prod_{\alpha=1}^{m}\delta(f_\alpha-g(r_\alpha)) . \quad (4)$$

A scaling homogeneous multifractal process is then a special case in which scaling exists between the measure and length scale $g(r)=r^D$. If in addition the fragmentation process is conservative ($\sum_{\alpha=1}^{m} r_\alpha^D=1$), then we have a homogeneous fractal process. For this special case, independent of the SPD and the averaging process, the generalized dimensions take the value D [see Eq. (21)]. However, we also need to consider the possibility of cases where small fragment ratios $r\to 0$ carry much less measure [for example, the stretched exponential form $g(r)\sim exp-\beta r^{-\gamma}$] or much more measure [for example, the logarithmic form $g(r)\sim 1/|\ln r|^{\lambda}$] than expected from scaling.

(ii) The extreme limit of weak correlation would be that no correlations exist between the measure fraction and the length scale ratio. Such an extreme limit seems unlikely on physical grounds. More realistically we can define independent fragmentation (IF) models in which, apart from the expected strong correlations, which must be expected between the measure and the length scale for a single fragment, the fragmentation probability is independent

$$P(f_1,\ldots,f_m;r_1,\ldots,r_m)=\prod_{\alpha=1}^{m}P(f_\alpha,r_\alpha) . \quad (5)$$

C. Conservation laws

During a fragmentation process conservation laws may strictly limit the domain in which the SPD is nonzero. Two important conservation laws are conservation of measure and conservation of volume.

(i) If measure is conserved or C process we expect

$$P(f_1,\ldots,f_m;r_1,\ldots,r_m)=Z^{-1}W(f_1,\ldots,f_m;r_1,\ldots,r_m)\delta\left[\sum_{\alpha=1}^{m}f_\alpha-1\right] . \quad (6)$$

(ii) If the process is space filling or S process we expect

$$P(f_1,\ldots,f_m;r_1,\ldots,r_m)=Z^{-1}W(f_1,\ldots,f_m;r_1,\ldots,r_m)\delta\left[\sum_{\alpha=1}^{m}r_\alpha^d-1\right] . \quad (7)$$

The characteristics defined above all constrain the form of an SPD. These characteristics are independent except for A and B processes, which are mutually exclusive. Therefore if we also allow as a separate characteristic the negation of these categories, e.g., nonconserved (NC) or nonhomogeneous, then it appears that we have defined $2^5=32$ mutually exclusive universality classes. It should be noted, however, that for real fragmentation process, we may only have partial information on some or several of these characteristics, e.g., we may only know that the fragmentation process is conservative. In this case, part of the experimental program would involve discovery of further categories which help refine the universality class to which the physical process belongs.

Thus the categories above can be combined to further specify a multiplicative process. For example, conservative independent stochastic fragmentation (CIF) processes will result in an SPD of the form

$$P(f_1,\ldots,f_m;r_1,\ldots,r_m)=Z^{-1}\prod_{\alpha=1}^{m}W(f_\alpha,r_\alpha)\delta\left[\sum_{\alpha=1}^{m}f_\alpha-1\right] . \quad (8)$$

Similarly space filling independent fragmentation (SIF) models will obey

$$P(f_1,\ldots,f_m;r_1,\ldots,r_m) = Z^{-1}\prod_{\alpha=1}^{m} W(f_\alpha,r_\alpha)\delta\left[\sum_{\alpha=1}^{m} r_\alpha^d - 1\right] . \quad (9)$$

One of the advantages about fragmentation which obeys any type of IF process [this includes Eqs. (5), (8), and (9)] is that the complete characterization of the stochastic process is given by the singularities of the probability $P(f,r)$ or weight $W(f,r)$ for a single measure fraction/length scale ratio fragment. They can be used to describe both AB and H processes depending on the structure of these single fragmentation probabilities or weights. Specifically for A processes $P(f,r)$ or $W(f,r)$ is nonzero between $r_{\min}\le r\le r_{\max}$ and $f_{\min}\le f\le f_{\max}$, while for B processes it is a smooth function in the complete integration domain $0\le r\le 1$ and $0\le f\le 1$, and if power law singularities exist, then these only occur at the domain edges

$$P(f,r),W(f,r)\sim\begin{cases} f^{\mu_0} & \text{as } f\to 0\\ (1-f)^{\mu_1} & \text{as } f\to 1\\ r^{\nu_0} & \text{as } r\to 0\\ (1-r)^{\nu_1} & \text{as } r\to 1 .\end{cases} \quad (10)$$

This assumption includes finite probabilities at the domain edges as a special case. These probability scaling exponents must obey $\mu,\nu>-1$ to allow for normalization. For H processes, on the other hand,

$$P(f,r)=P(r)\delta(f-g(r)) ,$$
$$W(f,r)=W(r)\delta(f-g(r)) , \quad (11)$$

and therefore, for example, an HIF process will have an SPD of the form

$$P(f_1,\ldots,f_m;r_1,\ldots,r_m)=\prod_{\alpha=1}^{m} P(r_\alpha)\delta(f_\alpha-g(r_\alpha)) . \quad (12)$$

One of the most important aspects of all these classes of models is that they are exactly renormalizable and the partition function for any individual member of an ensemble of stochastic processes can be explicitly calculated

$$Z_N(q,\tau)=\sum_{\{\alpha\}_N}\mu_\alpha^q/l_\alpha^\tau=(\mu_{\text{total}}^q/l_0^\tau)\sum_{\{\alpha\}_N}\prod_{i=1}^{N}[f_{\alpha_i}^q r_{\alpha_i}^{-\tau}]$$
$$=(\mu_{\text{total}}^q/l_0^\tau)\prod_{i=1}^{N}\left[\sum_{\alpha_i=1}^{m} f_{\alpha_i}^q r_{\alpha_i}^{-\tau}\right] , \quad (13)$$

and therefore so can the n replica average partition function

$$\langle\langle Z_N(q,\tau)^n\rangle\rangle=(\mu_{\text{total}}^{nq}/l_0^{n\tau})\left\langle\left\langle\prod_{i=1}^{N}\left[\sum_{\alpha_i=1}^{m} f_{\alpha_i}^q r_{\alpha_i}^{-\tau}\right]^n\right\rangle\right\rangle$$
$$=(\mu_{\text{total}}^{nq}/l_0^{n\tau})\left\langle\left[\sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau}\right]^n\right\rangle^N , \quad (14)$$

where $\langle\langle\ \rangle\rangle$ represents an average over all members of the stochastic ensemble, while $\langle\ \rangle$ represents an average over the SPD. Specifically for the annealed average

$$\langle\langle Z_N(q,\tau)\rangle\rangle=(\mu_{\text{total}}^q/l_0^\tau)\left\langle\left[\sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau}\right]\right\rangle^N , \quad (15)$$

while for the quenched average

$$\langle\langle \ln Z_N(q,\tau)\rangle\rangle=\ln(\mu_{\text{total}}^q/l_0^\tau)+N\left\langle\ln\left[\sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau}\right]\right\rangle . \quad (16)$$

Now as $N\to\infty$ the only way the partition function can be made to converge is if $\tau=\tau(q,n)$ such that

$$\left\langle\left[\sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau(q,n)}\right]^n\right\rangle=1 . \quad (17)$$

For the cases of annealed $(n=1)$ randomness, Eq. (17) simplifies to

$$\left\langle\left[\sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau_A(q)}\right]\right\rangle=1 , \quad (18)$$

while in the quenched $(n=0)$ case

$$\left\langle\ln\left[\sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau_Q(q)}\right]\right\rangle=0 . \quad (19)$$

We can also define generalized dimensions by

$$\tau(q,n)=(q-1)D(q,n) , \quad (20)$$

specifically annealed dimensions by $\tau_A(q)=(q-1)D_A(q)$, and quenched dimensions by $\tau_Q(q)=(q-1)D_Q(q)$. These dimensions are generalized Hausdorff dimensions rather than box counting dimensions, for in defining the partition function we are effectively covering the set at each stage by a set of boxes of different sizes $l\le\max l_\alpha$ and choosing τ uniquely such that the partition function neither diverges nor tends to zero.

Equation (17) is the basic generating equation which describes all members of this class of random weighted curdling models. Obviously its solution will vary with smooth variations of the SPD given by Eq. (3). In general such variations are nonuniversal, and there is no reason to believe that by using real data for the multifractal spectrum the SPD can be recovered. Underlying this trivial variation, however, deeper universal properties exist which can help us understand the physical properties of stochastic multifractal systems such as DLA and turbulence. We now proceed to investigate some of these universal properties.

III. FRACTAL AND INFORMATION DIMENSIONS

In this section we examine the general behavior to be expected for the Hausdorff dimensions $D(n)=D(q=0,n)$ and the information dimensions $D_I(n)=D(q=1,n)$ as a function of the averaging process.

Unless explicitly stated otherwise no specific assumptions about any singularities in the SPD are made here. Only for a homogeneous fractal processes with $g(r)=r^D$ is the dependence on q and the averaging process trivial. For this case

$$D(q,n)=D \ , \tag{21}$$

independent of q and the averaging process n. But this simple behavior is far from the generic dependence on these parameters to be expected for multifractals. Indeed for nonconservative processes with $\sum_{\alpha=1}^{m} f_\alpha < 1$ the information dimension does not even exist.

A. Hausdorff dimensions

As $q \to 0$ we see that Eq. (17) reduces to

$$\left\langle \left[\sum_{\alpha=1}^{m} r_\alpha^{D(n)} \right]^n \right\rangle = 1 \ . \tag{22}$$

For space filling processes the Hausdorff dimension is always the embedding dimension d. From an examination of Eq. (22) and the property of any S process that $\sum_{\alpha=1}^{m} r_\alpha^d = 1$, we immediately find the result

$$D(n)=d \ , \tag{23}$$

independent of the SPD and the replica average taken.

More generally for processes which are not space filling Eq. (22) reduces in the annealed case to

$$\left\langle \sum_{\alpha=1}^{m} r_\alpha^{D_A} \right\rangle = 1 \ , \tag{24}$$

while in the quenched case

$$\left\langle \ln \left[\sum_{\alpha=1}^{m} r_\alpha^{D_Q} \right] \right\rangle = 0 \ . \tag{25}$$

What is clear from Eqs. (23)–(25) is that, while for S processes the Hausdorff dimension is independent of the SPD and the averaging procedure, for processes confined to a fractal support, or non-space-filling (NS) processes, the measured Hausdorff dimensions will depend on the replica average n taken. This fact alone serves to distinguish experimentally two important classes of stochastic multifractal processes—S and NS—and would be a useful method to apply to inertial turbulence where the dimension of the dissipative field support remains in question.

B. Information dimensions

To find an expression for the information dimension $D_I(n)$ we need to take the limit $q \to 1$ of Eq. (17). A finite value for the information dimension only exists in the case of a C process. For this case if Eq. (17) has a solution, then we see that $\tau(q,n)>0$ for $q>1$, $\tau(q,n)=0$ for $q=1$, and $\tau(q,n)<0$ for $q<1$, and therefore the limit exists with $D_I(n)>0$. Indeed, for such a process the information dimension is independent of the replica average taken $D_I(n)=D_I$, with D_I the solution of the implicit equation $\langle \sum_{\alpha=1}^{m} f_\alpha \ln[f_\alpha r_\alpha^{-D_I}] \rangle = 0$, which has the explicit solution

$$D_I = \langle f \ln f \rangle / \langle f \ln r \rangle \ . \tag{26}$$

This is a remarkable result because not only is the information dimension independent of the replica average taken, it is also independent of the multiplicity of the splitting and dependent only on the probability $P(f,r)$ for a single splitting event. This universality does not apply to other generalized dimensions, which we study further in Secs. VI–VIII.

For NC processes, however, the information dimension does not exist, but rather $\tau(1,n)$ is finite and is given by

$$\left\langle \left[\sum_{\alpha=1}^{m} f_\alpha r_\alpha^{-\tau(1,n)} \right]^n \right\rangle = 1 \ . \tag{27}$$

Consequently for NC processes, the generalized dimension $D(q,n)$ treated as a function of the complex variable q has a simple pole on the real axis

$$D(q \to 1,n) \approx \tau(1,n)/(q-1) \ . \tag{28}$$

For example, for space filling but nonconservative homogeneous processes with $f=r^D$ we can calculate the generalized dimensions explicitly as $D(q,n)=(qD-d)/(q-1)$, independent of the replica average taken and the pole at $q=1$ is plainly visible.

Thus the existence of an information dimension serves to distinguish between conservative and nonconservative processes. Note that a combined use of the replica average dependence of the Hausdorff dimension with the behavior of the generalized dimension near $q=1$ already serves to separate four major universality classes (CS,CNS,NCS,NCNS) for stochastic multiplicative processes.

IV. STOCHASTIC MEASURE SINGULARITIES

The approach of characterizing multifractal measures as interwoven sets [45,46] of singularities with exponent α and associated fractal dimension $f(\alpha)$ is appropriate if on covering a multifractal of length scale L by boxes of size l the number of measure singularities of strength $\mu(l) \sim (l/L)^\alpha$ scale as $N(\alpha,l) \approx \rho(\alpha)(l/L)^{-f(\alpha)}$. This description is valid for deterministic multifractals, but has been applied mainly for characterizing stochastic systems. Yet it is clear that there is no such thing as a unique α versus $f(\alpha)$ curve for any stochastic multifractal. Rather each member of the ensemble must have its own unique distribution of singularities and any experimental plot must represent some average over the ensemble of such curves. Consequently the α versus $f(\alpha)$ description must be generalized to take this stochasticity into account.

To see how stochastic multiplicative distributions should be described, consider first the deterministic weighted curdling model in which at each stage of construction a line of length l_0 is split in two equal parts and the associated measure split into fractions f and $(1-f)$ on it. After N generations the length scale will be reduced to $l_N = l_0 2^{-N}$ and the measure will consist of 2^N fragments of which C_N^n have measure $f^n(1-f)^{N-n}$. In the appropriate limit ($N \to \infty, n \to \infty$, but $x=n/N$ finite) scaling exists with dimension $f(\alpha)$

$=-[x\ln x+(1-x)\ln(1-x)]/\ln 2$ for singularities $\alpha=-[x\ln f+(1-x)\ln(1-f)]/\ln 2$. Thus the α versus $f(\alpha)$ description is obeyed, both α and $f(\alpha)$ being parametrized by the single intensive variable x.

Now consider the case of a stochastic multiplicative process in which again the length scale is reduced by a factor of 2 at each generation, but this time with a probability p the measure splits into fractions f_1 and $(1-f_1)$, while with a probability $q=(1-p)$ it splits into fragments f_2 and $(1-f_2)$. We focus on a specific realization of this stochastic process after N generations in which the first possibility has occurred m times and the second $(N-m)$ times [let us call this the (m,N) realization]. This event will occur with a probability $P(m,N)=C_N^m p^m q^{N-m}$. In the intensive limit $(N\to\infty, m\to\infty$, but $c=m/N$ finite) this probability scales as $P(m,N)\to P(c)\sim(l_N/l_0)^{g(c)}$, where $g(c)=\{c\ln(c/p)+(1-c)\ln[(1-c)/q]\}/\ln 2$.

A specific fragment in this (m,N) realization of the multiplicative process is created by the measure fragmenting n_1 times by a factor f_1, and therefore $(m-n_1)$ by a factor $(1-f_1)$, and n_2 times by a factor f_2, and therefore $(N-m-n_2)$ times by a factor $(1-f_2)$. This singularity will have a measure $f_1^{n_1}(1-f_1)^{m-n_1}f_2^{n_2}(1-f_2)^{n-m-n_2}\sim(l_N/l_0)^{\alpha}$ associated with it. This singularity occurs with a frequency $C_m^{n_1}C_{N-m}^{n_2}$. For this stochastic case, scaling still exists in the intensive limit [with $n_1,n_2\to\infty$, but $x_1=n_1/m$ and $x_2=n_2/(N-m)$ finite], but now the dimension $f(x_1,x_2,c)=-\{c[x_1\ln x_1+(1-x_1)\ln(1-x_1)]+(1-c)[x_2\ln x_2+(1-x_2)\ln(1-x_2)]\}/\ln 2$ for singularity $\alpha(x_1,x_2,c)=-\{c[x_1\ln f_1+(1-x_1)\ln(1-f_1)]+(1-c)[x_2\ln f_2+(1-x_2)\ln(1-f_2)]\}/\ln 2$ depends on the realization considered.

This analysis leads us to the following ansatz for the distribution of singularities in stochastic multiplicative processes. A stochastic multiplicative process will generate an ensemble of multifractals which in the scaling limit can be described by a probability distribution

$$P(l,\mathbf{c})d\mathbf{c}\sim\rho_1(\mathbf{c})(l/L)^{g(\mathbf{c})}d\mathbf{c} \tag{29}$$

that a member of the ensemble when coarse grained to an inner scale l is described by a set of intensive order parameters $\mathbf{c}$ (in our case above c described the fraction of one set of splitting in a binary choice; a more complex fragmentation process requires more complex intensive descriptions).

If we define $\mathbf{x}$ to be a set of intensive descriptors of the singularities that may be generated [in the case above $\mathbf{x}=(x_1,x_2)$, the fraction of f_1 and f_2 splittings contributing to the singularity considered], then we may consider $\alpha(\mathbf{x},\mathbf{c})$ to be the singularity exponent

$$\mu(\mathbf{x},\mathbf{c},l)\sim(l/L)^{\alpha(\mathbf{x},\mathbf{c})} \tag{30}$$

and $f(\mathbf{x},\mathbf{c})$ the fractal dimension of this set related to the number of such singularities between $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$ by

$$N(\mathbf{x},\mathbf{c},l)d\mathbf{x}\sim\rho_2(\mathbf{x},\mathbf{c})(l/L)^{-f(\mathbf{x},\mathbf{c})}d\mathbf{x}\ . \tag{31}$$

Thus to calculate replica averages

$$\langle\langle Z^n\rangle\rangle=\left\langle\!\!\left\langle\left[\sum_i\mu_i(l)^q\right]^n\right\rangle\!\!\right\rangle =\int P(l,\mathbf{c})d\mathbf{c}\left[\int d\mathbf{x}\,N(\mathbf{x},\mathbf{c},l)\mu(\mathbf{x},\mathbf{c},l)^q\right]^n . \tag{32}$$

Asymptotically (provided $q\neq1$ and $n\neq0$) the inner integral has a maximum at $\mathbf{x}=\mathbf{x}(q,\mathbf{c})$ and using the saddle point method yields as $(l/L)\to0$

$$\int d\mathbf{x}\,N(\mathbf{x},\mathbf{c},l)\mu(\mathbf{x},\mathbf{c},l)^q\sim(l/L)^{-f[\mathbf{x}(q,\mathbf{c}),\mathbf{c}]+q\alpha[\mathbf{x}(q,\mathbf{c}),\mathbf{c}]}\ .$$

When this expression is substituted into Eq. (32) and the saddle point technique is applied again to the outer integral which maximizes at $\mathbf{c}=\mathbf{c}_{q,n}$ and $\mathbf{x}(q,\mathbf{c}=\mathbf{c}_{q,n})=\mathbf{x}_{q,n}$, a scaling form for the n replica average is found,

$$\langle Z^n\rangle\sim(l/L)^{g(\mathbf{c}_{q,n})+n[-f(\mathbf{x}_{q,n},\mathbf{c}_{q,n})+q\alpha(\mathbf{x}_{q,n},\mathbf{c}_{q,n})]}\ ,$$

and therefore

$$D(q,n)=g(\mathbf{c}_{q,n})/[n(q-1)] +[-f(\mathbf{x}_{q,n},\mathbf{c}_{q,n})+q\alpha(\mathbf{x}_{q,n},\mathbf{c}_{q,n})]/(q-1)\ . \tag{33}$$

For the case $q=1$, Eq. (32) can be used directly to find

$$D(1,n)=D_I =\int P(l,\mathbf{c})d\mathbf{c}\int d\mathbf{x}\,N(\mathbf{x},\mathbf{c},l)\mu(\mathbf{x},\mathbf{c},l)\alpha(\mathbf{x},\mathbf{c}) =\alpha(\mathbf{x}_{q=1,n=1},\mathbf{c}_{q=1,n=1})\ , \tag{34}$$

which in agreement with the generating function for the information dimension Eq. (26) is independent of n. The quenched limit $n\to0$ can also be found from Eq. (32)

$$(q-1)D(q,0) =\int P(l,\mathbf{c})d\mathbf{c}[-f(\mathbf{x}(q,\mathbf{c}),\mathbf{c})+q\alpha(\mathbf{x}(q,\mathbf{c}),\mathbf{c}) =[-f(\mathbf{x}_{q,n=0},\mathbf{c}_{n=0})+q\alpha(\mathbf{x}_{q,n=0},\mathbf{c}_{n=0})]\ . \tag{35}$$

The expression given by Eq. (33) differs in important respects from the expression derived originally by Halsey *et al.* [46] which remains valid for deterministic multifractals. Specifically a real α versus $f(\alpha)$ curve dominating all realizations of a stochastic multifractal does not exist in general, as if it did then the replica averaged partition function Eq. (32) would depend only trivially on n and $\tau(q,n)=\tau(q)=-f(\alpha_q)+q\alpha_q$. In the usual experimental situation, the procedure carried out normally involves calculating $\tau(q,n)$ and defining an effective α versus $f(\alpha)$ curve by $\tau(q,n)=-f(\alpha_{q,n})+q\alpha_{q,n}$. Clearly some type of averaged exponents are being found, but not the distribution of singularities directly generated by the stochastic multiplicative process, and this may hinder a proper understanding of the underlying physics.

V. INEQUALITIES AND ASYMPTOTICS

Universal relationships can be derived between generalized dimensions in the case of deterministic multiplicative processes in the form of inequalities [38]. In addition their asymptotic forms tend to be simple. In the case of stochastic processes the replica averaged exponents also have such universal behavior, their limiting form as $q \to \infty$ and $n \to \infty$ depending on robust properties of the SPD. Such inequalities and asymptotics can be expected to have a greater universality than the exact values for specific generalized dimensions (except for special results such as the exact result for the information dimension $D_I = 1$ for DLA in two dimensions).

A. Inequalities

If we have a set of random variables $a_i > 0$ and we define a mean for these variables using the rth moment as $m_r(a) = \langle a^r \rangle^{1/r}$, where $\langle \; \rangle$ represents an average over some distribution, then Hölder's inequality states that $m_r(a) > m_{r'}(a)$ if $r > r'$, except in the special case $P(a) = \delta(a - a_0)$ when all these means are the same $m_r(a) = a_0$. An important consequence of Hölder's inequality is that if we have the equality $m_r(a) = m_{r'}(a')$ for all $r > r'$, then this can only be true if $a_i' > a_i$ for all i. We make use of this corollary below.

Equation (17) in the case of conserved fragmentation (C process) can be rewritten in the form

$$M_{(q-1)n}[m_{q-1}(fr^{-D(q,n)})] = 1 \; , \tag{36}$$

where

$$m_{q-1}(fr^{-D(q,n)}) = \left[\sum_{\alpha=1}^{m} f_\alpha (f_\alpha r_\alpha^{-D(q,n)}) \right]^{1/(q-1)} ,$$

while $M_{(q-1)n}[a] = \langle a^{(q-1)n} \rangle^{1/[(q-1)n]}$.

First consider the case $q > q'$ at fixed n. We apply the corollary to Hölder's inequality twice: first, to the outer mean with the result $m_{q-1}(fr^{-D(q,n)}) > m_{q'-1}(fr^{-D(q',n)})$, and then for a second time to the inner mean with the result

$$D(q,n) \le D(q',n) \quad \text{if } q > q' \; . \tag{37}$$

This generalizes the known inequality $D(q) \le D(q')$ if $q > q'$ for deterministic fractals.

Now consider the case $n > n'$ at fixed q. In this case it will be useful to distinguish three regimes for q, namely, $q > 1$, $q = 1$, and $q < 1$. Again using Hölder's inequality, we find

$$\begin{aligned} &D(q,n) \ge D(q,n') \quad \text{if } n > n', \quad q < 1 \; , \\ &D(1,n) = D(1,n') \quad \text{if } n > n', \quad q = 1 \; , \\ &D(q,n) \le D(q,n') \quad \text{if } n > n', \quad q > 1 \; . \end{aligned} \tag{38}$$

For the case of nonconserved fragmentation (NC process) where $\sum_{\alpha=1}^{m} f_\alpha < 1$, Eq. (17) can be rewritten in the form

$$M_n \left[\left[\sum_{\alpha=1}^{m} f_\alpha \right] m_{q-1}(fr^{-D(q,n)})^{(q-1)} \right] = 1 \; , \tag{39}$$

which simplifies in the case that the total fraction of measure between generations is a fixed amount $\sum_{\alpha=1}^{m} f_\alpha = F < 1$ to

$$M_{(q-1)n}[m_{q-1}(fr^{-D(q,n)})] = F^{-1/(q-1)} \; . \tag{40}$$

We can see the singularity at $q = 1$, observed previously in Eq. (28); but for $q \ne 1$ the inequalitites found for conserved fragmentation Eqs. (37) and (38) still apply. If F is not fixed, but rather is a stochastic variable itself, then we are not able to derive these inequalities.

To find the generalization of the concave inequality $\tau(q_1+q_2) \ge [\tau(q_1)+\tau(q_2)]/2$ valid for deterministic multifractals, we use the Schwartz inequality in the form

$$\left[\sum_i \mu_i^{q_1+q_2} \right]^{2n} < \left[\sum_i \mu_i^{2q_1} \right]^n \left[\sum_i \mu_i^{2q_2} \right]^n , \tag{41}$$

which is valid for every individual member of a stochastic ensemble, and therefore when averaged over the ensemble implies

$$\begin{aligned} \left\langle\!\!\left\langle \left[\sum_i \mu_i^{q_1+q_2} \right]^{2n} \right\rangle\!\!\right\rangle &< \left\langle\!\!\left\langle \left[\sum_i \mu_i^{2q_1} \right]^n \left[\sum_i \mu_i^{2q_2} \right]^n \right\rangle\!\!\right\rangle \\ &\approx \left\langle\!\!\left\langle \left[\sum_i \mu_i^{2q_1} \right]^n \right\rangle\!\!\right\rangle \left\langle\!\!\left\langle \left[\sum_i \mu_i^{2q_2} \right]^n \right\rangle\!\!\right\rangle . \end{aligned} \tag{42}$$

In Eq. (42) we have assumed that it is permissible to neglect the correlations about their respective means of the two sums

$$\left\langle\!\!\left\langle \left[\left[\sum_i \mu_i^{2q_1} \right]^n - \left\langle\!\!\left\langle \left[\sum_i \mu_i^{2q_1} \right]^n \right\rangle\!\!\right\rangle \right] \times \left[\left[\sum_i \mu_i^{2q_2} \right]^n - \left\langle\!\!\left\langle \left[\sum_i \mu_i^{2q_2} \right]^n \right\rangle\!\!\right\rangle \right] \right\rangle\!\!\right\rangle \approx 0 \; .$$

Provided this assumption is valid, we find the generalized concave inequality

$$\tau(q_1+q_2, 2n) \ge [\tau(q_1,n)+\tau(q_2,n)]/2 \; . \tag{43}$$

Equation (43) reduces for quenched exponents to

$$(q_1+q_2-1)D_Q(q_1+q_2) \ge [(q_1-1)D_Q(q_1)+(q_2-1)D_Q(q_2)]/2 \; .$$

B. Asymptotic forms

The inequalities derived above fix qualitatively the shape of $D(q,n)$ as a function of q and n: for $q < 1$, $D(q,n)$ is a monotonically increasing function of n; for $q = 1$, the generalized dimension is independent of n, $D(1,n) = D_I$; for $q > 1$, $D(q,n)$ is a monotonically decreasing function of n. While at any n, $D(q,n)$ is a decreasing function of q. This still leaves open the questions of the limiting behavior as $q \to \infty$ for fixed n and as $n \to \infty$ for fixed q. To investigate these limits for A, B, and H processes let us now rewrite Eq. (17) explicitly as

$$\int \int P(f_1, \ldots, f_m; r_1, \ldots, r_m) \times \prod_\alpha df_\alpha \prod_\alpha dr_\alpha \left[\sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-(q-1)D(q,n)} \right]^n = 1 \tag{44}$$

and consider the implications as $q \to \infty$ for fixed n.

For type A processes, the main contribution to the integral comes from the corners of the $2m$ dimensional hypercube where $f_\alpha = f_{\max}$ and $r_\alpha = r_{\min}$. In this limit the integral can be estimated as $\approx C[f_{\max} r_{\min}^{-D(q,n)}]^{qn}/(qn)^2$, where C is a nonuniversal constant dependent of the assumed nonzero value for the probability on the domain boundary. Using this estimate then yields the asymptotic expression

$$\lim_{q\to\infty} D(q,n) \to \ln f_{\max}/\ln r_{\min} + O(\ln(qn)/(qn)) . \quad (45)$$

Thus for A processes a finite nonzero limit $D(\infty,n)$ exists which depends only on the largest measure and smallest length scale ratios into which fragmentation can occur.

For type B processes the $q\to\infty$ exponents will be controlled by the behavior of the probability distribution near $r=0$ and $f=1$

$$\lim_{q\to\infty} D(q,n) \to (1+\nu_0)/(qn) + O((qn)^{-(2+\mu_1)}) . \quad (46)$$

For H processes Eq. (44) reduces to

$$\int P(r_1,\ldots,r_m) \times \prod_\alpha dr_\alpha \left[\sum_{\alpha=1}^m g(r_\alpha)^q r_\alpha^{-(q-1)D(q,n)}\right]^n = 1 . \quad (47)$$

The nature of the asymptotic behavior is controlled by both $g(r)$ and $P(r_1,\ldots,r_m)$. Suppose, for example, that as for an A process the domain in which the probability is nonzero lies between $r_{\min} < r_\alpha < r_{\max}$ and $g(r)$ is a monotonically increasing function of r. Then the most important contribution to the integral lies at the domain edge $r_\alpha = r_{\max}$ and as $q\to\infty$ we find

$$\lim_{q\to\infty} D(q,n) = > \ln g(r_{\max})/\ln r_{\max} , \quad (48)$$

independent of the multiplicity and averaging process.

Let us now consider the asymptotic behavior as $n\to\infty$ for fixed q. For type A processes we need to solve

$$\left[\sum_{\alpha=1}^m f_\alpha^q r_\alpha^{-(q-1)D(q,\infty)}\right]_{\max} = 1 . \quad (49)$$

In other words, we need to find the value of $D(q,\infty)$ such that the maximum value of the left-hand side of Eq. (49) over all of the domain where $P(f_1,\ldots,f_m;r_1,\ldots,r_m) \neq 0$ yields a value of 1. Note that this value is universal in the sense that it does not depend on the form of the probability distribution where it is nonzero.

For $q>1$, the maximum lies at $r_{\min}$; for $q<1$ the maximum occurs at $r_{\max}$. In addition the sum $[\sum_{\alpha=1}^m f_\alpha^q]_{\max} \approx f_{\max}^q$ for q large enough; for $q<1$ the maximum occurs at $f_\alpha = 1/m$ for C processes and therefore $[\sum_{\alpha=1}^m f_\alpha^q]_{\max} = m^{1-q}$. Putting these results together yields the finite limits for the generalized dimensions $\lim_{n\to\infty} D(q,n) \to D(q,\infty)$ where

$$D(q,\infty) = \begin{cases} -\ln m/\ln r_{\max} & \text{if } q<1 \\ \ln\left[\sum_{\alpha=1}^m f_\alpha^q\right]_{\max} \Big/ [(q-1)\ln r_{\min}] , & \text{if } q>1 \end{cases} \quad (50)$$

which for $q \gg 1$ can be estimated as $[q/(q-1)]\ln f_{\max}/\ln r_{\min}$. Notice that as $r_{\max}\to 1$, $D(q,n)\to\infty$ for $q<1$; as $r_{\min}\to 0, D(q,n)\to 0$ for $q>1$. It appears therefore that there is nothing stopping a generalized dimension from having larger values than the embedding dimension.

For B processes, these finite limits do not exist and we rather have to estimate $D(q,n\to\infty)$ from Eqs. (44) and (10) directly. We find in this case

$$\lim_{n\to\infty} D(q,n) \to \begin{cases} m^{(1-q)n/(1+\nu_1)}/[(1-q)n] & \text{if } q<1 \\ (1+\nu_0)/[(q-1)n] + O(n^{-(2+\mu_1)}) & \text{if } q>1 . \end{cases} \quad (51)$$

For H processes Eq. (49) reduces to

$$\left[\sum_{\alpha=1}^m g(r_\alpha)^q r_\alpha^{-(q-1)D(q,\infty)}\right]_{\max} = 1 . \quad (52)$$

Thus $D(q,\infty)$ is controlled solely by $g(r)$ and given by

$$D(q,\infty) \to [q/(q-1)] d\ln g[r(q)]/d\ln r(q) , \quad (53)$$

where $r(q)$ is defined by the implicit equation $mg[r(q)]^q r(q)^{-(q-1)D(q,\infty)} = 1$. Thus it is possible to distinguish between type A and B processes based on whether their asymptotic generalized dimensions tend to zero or not as $q\to\infty$ and their behavior as $n\to\infty$ for both $q<1$ and $q>1$.

VI. INDEPENDENT STOCHASTIC FRAGMENTATION

In this section we shall assume that the m fold splitting process at any given generation can be regarded as a set of m independent events. In other words, we shall consider the NCIF class of models given by the distribution

$$P(f_1,\ldots,f_m;r_1,\ldots,r_m)\prod_\alpha df_\alpha \prod_\alpha dr_\alpha = \prod_{\alpha=1}^m [P(f_\alpha,r_\alpha) df_\alpha dr_\alpha] . \quad (54)$$

This universality class of fragmentation processes keeps the physically plausible strong correlations to be expected between the random variables (f_α, r_α), but loses the possibility of describing exactly physical processes involving either strict measure conservation $\sum_{\alpha=1}^m f_\alpha = 1$ or strict space filling $\sum_{\alpha=1}^m r_\alpha^d = 1$. Both of these conservation laws can, however, be incorporated into NCIF models on the average by fixing the appropriate first moments $\langle f\rangle = 1/m$ in the case of measure conservation and by $\langle r^d\rangle = 1/m$ for space filling processes.

We can write Eq. (17) for this class of processes as

$$\int \prod_{\alpha=1}^m [P(f_\alpha,r_\alpha) df_\alpha dr_\alpha] \left[\sum_{\alpha=1}^m f_\alpha^q r_\alpha^{-\tau(q,n)}\right]^n = 1 , \quad (55)$$

which for integral n can be expanded in the form

$$\sum_{\alpha_1=1}^{m} \cdots \sum_{\alpha_n=1}^{m} \int \prod_{\alpha=1}^{m} [P(f_\alpha,r_\alpha)df_\alpha dr_\alpha] \\ f_{\alpha_1}^q r_{\alpha_1}^{-\tau(q,n)} \cdots f_{\alpha_n}^q r_{\alpha_n}^{-\tau(q,n)} = 1 \ . \quad (56)$$

Let us use the notation n_α to mean the number of subscripts with a given $1 \le \alpha \le m$. Then we can rewrite Eq. (56) as

$$\sum_{n_1=0}^{n} \cdots \sum_{n_m=0}^{n} \delta_{n,\sum_\alpha n_\alpha} n!/[n_1! \cdots n_m!] \\ \times \prod_{\alpha=1}^{m} \langle [f_\alpha^q r_\alpha^{-\tau(q,n)}]^{n_\alpha} \rangle = 1 \ , \quad (57)$$

where $\langle \ \rangle = \int \int P(f,r)df\,dr$. Finally using the identity $\delta_{n,m} = (1/2\pi)\int_0^{2\pi} \exp[i\theta(m-n)]d\theta$ for the Kronecker delta function, we find

$$(n!/2\pi)\int_0^{2\pi} d\theta \exp(-in\theta)\langle g_n(f^q r^{-\tau(q,n)} \exp i\theta)\rangle^m = 1 \ , \quad (58)$$

where we have used the notation $g_n(t) = \sum_{k=0}^{n} t^k/k!$. Thus for NCIF models one can reduce the generating function to a form dependent on only the scaling probability distribution for a single independent splitting at any generation. Equation (58) can also be rewritten as a closed contour integral about the unit circle in the complex plane by defining $z = \exp(i\theta)$ in which case we have

$$(n!/2\pi i)\oint (dz/z^{1+n})\langle g_n(f^q r^{-\tau(q,n)} z)\rangle^m = 1 \ . \quad (59)$$

For integer n there is an $(n+1)$th-order pole at $z=0$, and therefore using the calculus of residues and assuming no other singularities in the unit circle Eq. (59) reduces to

$$d^n \langle g_n(f^q r^{-\tau(q,n)} z)\rangle^m / dz^n |_{z=0} = 1 \ . \quad (60)$$

For annealed exponents Eq. (60) becomes

$$\langle f^q r^{-\tau_A(q)} \rangle = 1/m \ , \quad (61)$$

and therefore the annealed Hausdorff dimension obeys $\langle r^{D_A} \rangle = 1/m$ in NCIF models, while in the $q \to 1$ limit the information dimension is consistent with Eq. (26), provided the NCIF process is conservative on the average.

Other simplifications occur as $n \to \infty$ when Eq. (60) reduces to $d^n \langle \exp f^q r^{-\tau(q,n)} z \rangle^m / dz^n |_{z=0} = 1$, while for $n=2$, the generalized dimensions can be calculated from the generating function $m\langle (f^q r^{-\tau(q,2)})^2 \rangle + m(m-1)\langle f^q r^{-\tau(q,2)} \rangle^2 = 1$. Notice that all these forms display the explicit dependence on multiplicity m of the multifractal spectrum in contrast with Eq. (17).

To derive the quenched exponents, however, we need to consider analytic continuation of Eq. (55) as $n \to 0$. In these investigations multifractal dispersion relations play a significant role.

VII. QUENCHED AVERAGES AND ANALYTIC CONTINUATION

From an experimental viewpoint, quenched exponents are the least subject to poor statistics due to undersampling the stochastic ensemble, and therefore the most accurate in comparison with theoretical predictions.

To analyze the $n \to 0$ limit of NCIF models we need to analytically continue Eq. (55), which can best be done using the Dirac delta function representation

$$\int \prod_{\alpha=1}^{m} [P(f_\alpha,r_\alpha)df_\alpha dr_\alpha] \left[\sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau} \right]^n \\ = \int_0^\infty dx\, x^n \int \prod_{\alpha=1}^{m} [P(f_\alpha,r_\alpha)df_\alpha dr_\alpha] \\ \times \delta \left[x - \sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau} \right] . \quad (62)$$

The limits of the integral in Eq. (62) need only be taken over positive x values as $\sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau} > 0$ for all members of the stochastic ensemble. On using the identity $\delta(x) = 1/(2\pi)\int_{-\infty}^{+\infty} dk \exp ikx$, Eq. (62) can be rewritten

$$\int_0^\infty dx\, x^n \int \prod_{\alpha=1}^{m} [P(f_\alpha,r_\alpha)df_\alpha dr_\alpha] \delta \left[x - \sum_{\alpha=1}^{m} f_\alpha^q r_\alpha^{-\tau} \right] \\ = \int_{-\infty}^{+\infty} [dk/(2\pi)] G(q,\tau,m;-k) h(k,n) \ . \quad (63)$$

Here

$$h(k,n) = \int_0^\infty dx\, x^n \exp(ikx) \\ = \exp[i\pi(n+1)/2\, \mathrm{sgn}(k)] n!/|k|^{1+n} \ , \quad (64)$$

where $\mathrm{sgn}(k)$ means the sign of k, $h(k,n)$ is the Fourier transform of the generalized function $|x|^n[1+\mathrm{sgn}(x)]/2$ (see, for example, [60]), and $G(q,\tau,m;k)$ is the characteristic function

$$G(q,\tau,m;k) = \langle \exp(ikf^q r^{-\tau}) \rangle^m \ , \quad (65)$$

where as before $\langle \ \rangle = \int \int P(f,r)df\,dr$.

We note that if we analytically continue the characteristic function into the complex plane, then the resulting function is analytic in the upper half complex plane $\mathrm{Im}k > 0$. Therefore Titchmarsh's theorem applies, and we may write the dispersion relations

$$\mathrm{Re}G(q,\tau,m;k') \\ = P(1/\pi)\int_{-\infty}^{+\infty} dk\, \mathrm{Im}G(q,\tau,m;k)/(k-k') \ , \\ \mathrm{Im}G(q,\tau,m;k') \\ = -P(1/\pi)\int_{-\infty}^{+\infty} dk\, \mathrm{Re}G(q,\tau,m;k)/(k-k') \ , \quad (66)$$

where P means the principal part of the integral. These dispersion relations can be used to generate several useful identities including $P(1/\pi)\int_{-\infty}^{+\infty} dk\, \mathrm{Im}G(q,\tau,m;k)/k = 1$ and $P(1/\pi)\int_{-\infty}^{+\infty} dk\, \mathrm{Re}G(q,\tau,m;k)/k = 0$. Other useful properties of the characteristic function include $G(q,\tau,m;0) = 1$ and $G(q,\tau,m;-k) = G(q,\tau,m;k)^*$.

The representation for the generating function given by Eq. (63) is useful because the characteristic function $G(q,\tau,m;k)$ does not depend on the replica average taken and its multiplicity m dependence is very simple, while the n dependence in $h(k,n)$ is valid for nonintegral n.

Also note that in Eq. (63) it is $G(q,\tau,m;-k)$ that appears inside the integral and not $G(q,\tau,m;k)$.

Because of the singularity in $h(k,n)\sim|k|^{1+n}$ as $k\rightarrow 0$, Eq. (63) is an improper integral for $n\geq 0$ and to analytically continue into this domain we integrate by parts with the result that for $n>-1$ we find

$$\int_{-\infty}^{+\infty}[dk/(2\pi)]G(q,\tau,m;-k)h(k,n)=[\Gamma(n)/\pi]\int_0^{\infty}(dk/k^n)(d/dk)\{\mathrm{Im}G(q,\tau,m;k)\cos(\pi n/2) - \mathrm{Re}G(q,\tau,m;k)\sin(\pi n/2)\} . \quad (67)$$

It is now possible to take the $n\rightarrow 0$ limit, with the final result that Eq. (55) reduces to the implicit equation for the quenched multifractal spectrum $\tau_Q(q)$,

$$\int_0^{\infty} dk\, \ln k\,[d\,\mathrm{Re}G(q,\tau_Q(q),m;k)/dk - (2/\pi)\mathrm{Im}G(q,\tau_Q(q),m;k)/k]=2C , \quad (68)$$

where $C=0.577\,215\,66\ldots$ is the Euler-Masheroni constant. In deriving Eq. (68), use was made of the identity

$$\int_0^{\infty} dk\, \ln k d\, \mathrm{Im}G(q,\tau_Q(q),m;k)/dk=-\pi/2 ,$$

which can be derived from the multifractal dispersion relations.

VIII. PHASE TRANSITIONS AND MINIMAL PROBABILITIES

For a multifractal with no phase transitions the complete spectrum of $D(q,n)$ exists between $-\infty<q<\infty$. But in many cases the structure of the SPD will result in the existence of phase transitions, which imply that $D(q,n)$ is only defined for $q_{\mathrm{bottom}}(n)<q<q_{\mathrm{top}}(n)$. Among the most robust and universal properties of stochastic multifractal spectra are the position and nature of singularities at these phase transitions. They appear to be dependent only on the nature of singularities in the SPD and on the replica average taken. This observation implies that a great deal of useful information on the multiplicative random process controlling stochastic multifractals can be extracted from studies of $q_{\mathrm{bottom}}(n)$ and $q_{\mathrm{top}}(n)$, the positions where multifractal phase transitions occur as functions of the replica average n used, and the nature of any singularities in $D(q,n)$ close to these points.

Let us first consider a type A process. Because the region of (f,r) integration is bounded for such a process, all the moments in Eq. (69) exist for all finite q. As a consequence, type A processes have a complete multifractal spectrum with no phase transition whatever the shape of the SPD (naturally assuming that no singularities exist in the bounded integration domain).

This situation changes drastically for type B processes. Starting with Eq. (17) in the expanded form

$$\sum_{n_1\cdots n_m}(n!/[n_1!\cdots n_m!])\left[\prod_{\alpha=1}^{n} f_\alpha^{qn_\alpha}\prod_{\alpha=1}^{m} r_\alpha^{-\tau(q,n)n_\alpha}\right]=1 , \quad (69)$$

where the sum is over terms obeying $\sum_\alpha n_\alpha=n$, we note that the different moments contributing to the sum above are more or less singular depending on q and the moment considered. The fact that a breakdown in scaling can be expected in Eq. (69) can be seen by considering the effect of measure singularities as $f\rightarrow 0$. For $q>0$ the most singular behavior occurs for moments with $n_\alpha=0$, and as $P(f,r)\sim f^{\mu_0}$ we note that all the moments exist provided $\mu_0>-1$. For $q<0$, however, the most singular behavior occurs for moments with $n_\alpha=n$; in this case all the moments only exist provided

$$q>q_{\mathrm{bottom}}(n)=-(\mu_0+1)/n . \quad (70)$$

Specifically $q_{\mathrm{bottom},A}=-(\mu_0+1)$ for annealed averages, while $q_{\mathrm{bottom},Q}=-\infty$ for quenched averages.

Indeed divergent behavior in $D(q,n)$ can be expected as $q\rightarrow q_{\mathrm{bottom}}(n)$ for B processes and the generalized dimension is increasingly influenced by the set of very weak measure fragments at $f\rightarrow 0$ and $r\rightarrow 1$. We find

$$D(q,n)\sim -(\mu_0+qn+1)^{-1/(\nu_1+1)}/[(q-1)n] \quad (71)$$

as $q\rightarrow q_{\mathrm{bottom}}(n)=-(\mu_0+1)/n$. This singular behavior is universal, depending as it does only on q, n, and the singularities in the SPD associated with the weakest fragments.

To analyze $q_{\mathrm{top}}(n)$ we note that for $q>1$ the most divergent integral occurs with $n_\alpha=n$. As f and r are bounded from above, however, and $D(q,n)$ can always be chosen so that there are no divergences in the moments as $r\rightarrow 0$ provided $\nu_0>-1$ and $n>0$ we find

$$q_{\mathrm{top}}(n)=\infty . \quad (72)$$

These expressions are independent of the splitting multiplicity m and this is confirmed by an examination of the exact results for the NCIF model for which we need only examine the conditions for the existence of the average $\langle g_n(f^q r^{-\tau}z)\rangle$ in Eq. (59), which again is independent of m.

The results above are only valid for $\mu_0>-1$ and $\nu_0>-1$. In this regime the central limit theorem holds, in consequence of which the random variables $(1/N)\ln[\mu_\alpha/\mu_{\mathrm{total}}]$ and $1/N\ln[l_\alpha/l_0]$ have lognormal distributions. For $\mu_0<-1$ and $\nu_0<-1$, on the other hand, the distributions cannot be normalized. This still leaves open the behavior of an important class of distributions of the form

$$P_1(f,r)\sim 1/[f|\ln f|^{\lambda_1}] ,$$
$$P_2(f,r)\sim 1/[r|\ln r|^{\lambda_2}] \quad (73)$$

as $f\rightarrow 0$ and $r\rightarrow 0$, respectively. For these distributions to be normalizable we require that the measure exponent $\lambda_1>1$ and the same for the length scale exponent $\lambda_2>1$. An examination of the second moments $\langle(\ln f)^2\rangle$ and $\langle(\ln r)^2\rangle$ shows that for $\lambda_1>3$ and $\lambda_2>3$ the central limit still applies, but for smaller values a breakdown of log normality can be expected. For this distribution a multifractal phase transition can be expected at

$$q_{\rm bottom}(n)=0 \quad \text{if } \mu_0=-1 \ , \qquad q_{\rm top}(n)=1 \quad \text{if } \nu_0=-1 \tag{74}$$

for all $n>0$, and therefore an upper critical value for q exists in this case.

For quenched exponents $n=0$ more care needs to be taken, and this can be done by examining the conditions for the existence of $\langle \ln[\sum_{\alpha=1}^{m} f_\alpha^{-\tau_Q(q)}]\rangle$. Three ranges emerge: (a) for $q>1$, this average exists provided $\lambda_1>1$ and $\lambda_2>2$; (b) for $0<q<1$, the average exists for all values of $\lambda_1>1$ and $\lambda_2>1$; (c) for $q<0$, the average exists provided $\lambda_1>2$ and $\lambda_2>1$. These results imply that for quenched exponents the bottom end of the multifractal spectrum obeys

$$q_{{\rm bottom},Q}=\begin{cases}-\infty & \text{if } \mu_0>-1 \ \text{ or } \mu_0=-1, \ \lambda_1\ge 2\\ 0 \ \text{ if } \mu_0=-1, \ \lambda_2\le 2\end{cases} \tag{75}$$

while the top end obeys

$$q_{{\rm top},Q}=\begin{cases}\infty & \text{if } \nu_0>-1 \ \text{ or } \nu_0=-1, \ \lambda_2\ge 2\\ 1 \ \text{ if } \nu_0=-1, \ \lambda_2\le 2 \ .\end{cases} \tag{76}$$

For H processes an examination of Eq. (47) shows that $q_{\rm bottom}(n)$ is controlled by the behavior of $g(r)$. If a finite $r_{\rm min}$ exists, then no multifractal phase transitions will occur. If, however, $g(r)$ is singular as $r\to 0$, then a finite $q_{\rm bottom}(n)$ may be expected. For example, if $g(r)\sim\exp(-\beta r^{-\gamma})$ has a stretched exponential form, $q_{\rm bottom}(n)=0$ independent of the averaging process.

For such stretched exponential forms for $g(r)$ as $q\to 0^+$, $D(q,n)\to D(n)$, the finite ensemble averaged fractal dimension obeying

$$\int P(r_1,\ldots,r_m)\prod_\alpha dr_\alpha\left[\sum_{\alpha=1}^{m} r_\alpha^{D(n)}\right]^n=1 \ . \tag{77}$$

For small q, near the multifractal phase transition, singularities of the form $D(q,n)\approx D(n)+q^{\lambda(n)}f(n)$ may occur in the generalized dimension depending on the nature of the averaging process and the stretched exponential exponent γ. Indeed, this appears to be the case. We find

$$\lambda(n)=\begin{cases}1 & \text{if } \gamma<D(n)+1\\ [D(n)+1]/\gamma & \text{if } \gamma>D(n)+1 \ .\end{cases} \tag{78}$$

Closely connected to the existence of multifractal phase transitions is the scaling behavior of the minimal measure fragments $\mu_{\rm min}(\mathcal{N})$ with the number of fragments $\mathcal{N}$. Consider a single realization of a fragmentation process of multiplicity m, which has occurred N times. The total number of fragments is $\mathcal{N}=m^N$. What is the minimal measure that is likely to be observed? This measure can be estimated as

$$\mu_{\rm min}(\mathcal{N})\sim\mu_{\rm total}f_{\rm min}(\mathcal{N})^N \ , \tag{79}$$

where $f_{\rm min}(\mathcal{N})$ is the minimal measure fraction likely to occur during N multiplicative events and $\mu_{\rm total}$ here represents the initial total measure.

For A processes $f_{\rm min}$ is simply the lower edge of the measure domain of integration and therefore the minimal measure decreases as a power law in the number of fragments

$$\mu_{\rm min}(\mathcal{N})\sim\mu_{\rm total}\mathcal{N}^{-|\ln f_{\rm min}|/\ln m} \ . \tag{80}$$

For B processes, we can estimate $f_{\rm min}(\mathcal{N})$ from the equation $\mathcal{N}[\int_0^{f_{\rm min}}df\,P(f)]\sim 1$, where $P(f)=\int_0^1 dr\,P(f,r)$. Thus the scaling behavior of the SPD as $f\to 0$ will control the minimal observed fragment and we find $f_{\rm min}(\mathcal{N})\sim\mathcal{N}^{-1/(\mu_0+1)}$ for $\mu_0>-1$. Combining this estimate with Eq. (79) yields a log normal (in the number of fragments) form for the minimal measure

$$\mu_{\rm min}(\mathcal{N})\sim\mu_{\rm total}\exp\{-(\ln\mathcal{N})^2/[(\mu_0+1)\ln m]\} \ . \tag{81}$$

If the central limit theorem breaks down and the SPD has the form given by Eq. (73), then we find that $f_{\rm min}(\mathcal{N})\sim\exp[-c\mathcal{N}^{1/(\lambda_1-1)}]$. The minimal measure for this case has a stretched exponential form

$$\mu_{\rm min}(\mathcal{N})\sim\mu_{\rm total}\exp[-c\mathcal{N}^{1/(\lambda_1-1)}\ln\mathcal{N}/\ln m] \ , \tag{82}$$

where c is some constant.

Finally, for H processes $\int_0^{f_{\rm min}(\mathcal{N})}df\,P(f)=\int_0^{r_{\rm min}}dr\,P(r)$, where $f_{\rm min}(\mathcal{N})=g(r_{\rm min})$ and therefore in this case the minimal fraction behavior depends on the short length scale behavior of $g(r)$. If we assume the stretched exponential form $g(r)\sim\exp-\beta r^{-\gamma}$, we find that $f_{\rm min}(\mathcal{N})\sim(\ln\mathcal{N})^{(\gamma+1)/\gamma}/\mathcal{N}$, and therefore once more we find a scaling form for the minimal measure which is very close to log normal in the number of fragments

$$\mu_{\rm min}(\mathcal{N})\sim\mu_{\rm total}\exp\{-[(\gamma+1)/\gamma](\ln\mathcal{N})^2\ln\ln\mathcal{N}/\ln m\} \ . \tag{83}$$

In summary, the existence of multifractal phase transitions is associated with minimal measure fragments which tend to zero faster than any power law in the number of fragments; the breakdown in log normal behavior and the central limit theorem for the random measures are associated with stretched exponential forms for the minimal measures.

IX. BINARY CONSERVED FRAGMENTATION

In this section we study conserved CIF, especially binary conserved fragmentation. This is perhaps the physically most important of stochastic multiplicative processes for it has been be used to describe both inertial turbulence and, as we shall explicitly demonstrate, DLA. The ansatz for the SPD is taken to be

$$P(f_1,\ldots,f_m;r_1,\ldots,r_m)=Z^{-1}\prod_{\alpha=1}^{m}W(f_\alpha,r_\alpha)\delta\left[\sum_{\alpha=1}^{m}f_\alpha-1\right] \ , \tag{84}$$

where the partition function Z is given by

$$Z=\int\delta\left[\sum_{\alpha=1}^{m}f_\alpha-1\right]\prod_{\alpha=1}^{m}[W(f_\alpha,r_\alpha)df_\alpha dr_\alpha]\ . \quad (85)$$

The first thing to notice about Eq. (84) is that the equalities involving the generating functions for a CIF process can be explicitly written down by a simple transformation of those for NCIF models by using the integral representation of the Dirac delta function $\delta(x)=(1/2\pi)\int_{-\infty}^{+\infty}\exp(ikx)dk$ to replace any nonconserved fragmentation average of the form $\langle X\rangle^m=[\int\int P(f,r)X(f,r)df\,dr]^m$, where m is the splitting multiplicity, by the expression

$$Z^{-1}(1/2\pi)\int_{-\infty}^{+\infty}dk\,\exp(-ik)\times\left[\int\int W(f,r)\exp[ikf]X(f,r)df\,dr\right]^m .$$

(In fact the same procedure can be applied to SIF processes where

$$Z^{-1}(1/2\pi)\int_{-\infty}^{+\infty}dk\,\exp(-ik)\times\left[\int\int W(f,r)\exp[ikr^d]X(f,r)df\,dr\right]^m$$

is the proper transformation of the nonconserved average.)

For many physical processes binary fragmentation $m=2$ is either an exact (we consider binary DLA in Sec. X) or a reasonable approximation (Meneveau and Sreenivasan [59] have argued that a binomial cascade model suffices to account for the multifractal spectrum of inertial turbulence). Also this binary cascade model maximizes correlations between the appearance of small and large measure fractions and therefore any singularities in these limits have an enhanced effect on the spectral behavior.

For conserved binary fragmentation Eq. (17) can be rewritten as

$$Z^{-1}\int df\int dr_1\int dr_2 W(f,r_1)W(1-f,r_2)\times[f^q r_1^{-(q-1)D(q,n)}+(1-f)^q r_2^{-(q-1)/D(q,n)}]^n=1\ , \quad (86)$$

where $Z=\int df\,\hat{W}(f)\hat{W}(1-f)$ and $\hat{W}(f)=\int dr\,W(f,r)$. As a consequence, the probability density $P(f,r)$ of finding an individual fragmentation piece of measure fraction f and length scale ratio r is given by

$$P(f,r)=Z^{-1}W(f,r)\hat{W}(1-f)\ , \quad (87)$$

and therefore the probability $P(f)$ of finding a measure fragment f whatever its length scale is

$$P(f)=Z^{-1}\hat{W}(f)\hat{W}(1-f)\ , \quad (88)$$

which is a symmetric function about $f=\frac{1}{2}$. Similarly the probability $P(r)$ of observing a fragment of any given length scale r independent of its associated measure is

$$P(r)=\int df\,P(f,r)=Z^{-1}\int df\,W(f,r)\hat{W}(1-f)\ . \quad (89)$$

In contrast to Eq. (88), $P(r)$ will exhibit no special symmetry properties in general.

If the binary CIF process has an A form for the weight $W(f,r)$, no singular behavior will occur in $P(f,r)$ or $P(f)$. Scaling may appear in these reduced probability distributions if singularities in the weights $W(f,r)$ exist. For B processes, the weights will be dominated by singularities at the boundary of the (f,r) domain. As a consequence one would expect singular behavior in $P(f,r)$ of the form

$$P(f,r)\sim\begin{cases}f^\mu & \text{as } f\to 0\\ (1-f)^\mu & \text{as } f\to 1\ ,\end{cases} \quad (90)$$

where $\mu=\mu_0+\mu_1$ [see Eq. (10)]. This symmetric but more singular behavior is a manifestation of the maximal correlation between the appearance of small and large fragments in binary conserved processes. Similarly the exponent μ also replaces μ_0 in Eq. (70) for the position of multifractal phase transitions.

A second alternative is that no power law singularities appear in the weight $W(f,r)$ at the domain edge, but rather we are dealing with an H process. For this case $W(f,r)=W(r)\delta(f-g(r))$ [see Eq. (11)]. One consequence of the scaling ansatz $g(r)=r^D$ is that $P(r)$, while not a symmetric function of r about $r=\frac{1}{2}$, does obey the form

$$P(r)=W(r)W((1-r^D)^{1/D})(1-r^D)^{(1-D)/D}\ , \quad (91)$$

and therefore the function $P(r)(1-r^D)^{(D-1)/D}$ should be a symmetric function of r^D about $r^D=\frac{1}{2}$. Also the weight $\hat{W}(f)$ for such scaling H processes obeys

$$\hat{W}(f)=W(f^{1/D})f^{(1-D)/D}\ . \quad (92)$$

The reduced weight $W(r)$ may also show scaling behavior $W(r)\sim r^\nu$ as $r\to 0$, while $W(r)\sim(1-r)^{\nu'}$ as $r\to 1$, in which case then we shall call such scaling behavior a corner singularity as it implies a singular behavior in the SPD at the corners ($r=0$, $f=0$ or $r=1$, $f=1$) of its domain. In conclusion, depending on the dominant process involved we may expect different types of scaling for $P(f)$

$$P(f)\sim\begin{cases}f^{\mu_0+\mu_1} & \text{for B processes}\\ f^{(\nu+D\nu'+1-D)/D} & \text{for scaling H processes}\end{cases} \quad (93)$$

and such scaling is indeed observed in DLA.

X. MASS AND GROWTH DISTRIBUTIONS FOR DLA

As a physical example of the use of SPDs for characterizing stochastic multifractals we shall consider diffusion-limited aggregation. DLA can be characterized as a C fragmentation process. We associate a measure μ_{branch} of interest with branches of the DLA cluster. For example, here we consider the mass measure (or number of particles) in branches and as a second example the probability that random walks stick to a given branch. As each branch consists of a node and m subbranches, and both mass and growth probability are conservative, we can split the measures into fractions

$f_\alpha = \mu_{\text{subbranch},\alpha}/\mu_{\text{branch}}$, where $\mu_{\text{subbranch},\alpha}$ is the measure associated with each subbranch α. We can also associate a length scale l_{branch} with each branch, for example, its radius of gyration or the maximal diameter of the set of particles in a branch. The ratio of length scales between a branch of length scale l_{branch} and its subbranches $l_{\text{subbranch},\alpha}$ defines the set of ratios $r_\alpha = l_{\text{subbranch},\alpha}/l_{\text{branch}}$.

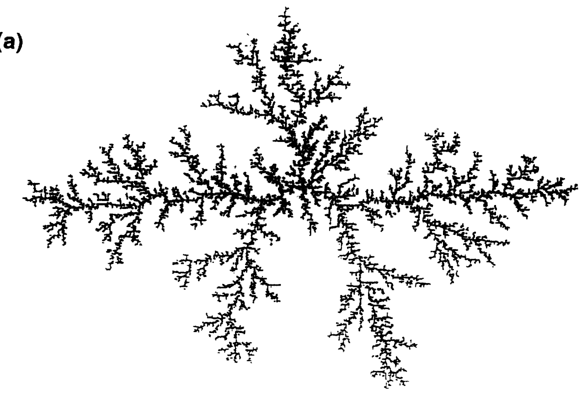
Because the choice of length scale is not unique, the SPD is to some degree dependent of the choice made. For example, if the maximal diameter is chosen as the length scale l_{branch}, then $r \le 1$, but if both particles defining this diameter lie in a single subbranch, then $r = 1$ for this fragment and therefore we might expect sharp singularity to exist on the boundary $P(f, r=1)$. On the other hand, if the radius of gyration is chosen as the length scale, then $r > 1$ is possible (consider a branch consisting of two subbranches, one of which is essentially linear and one which is almost compact—the linear fragment can be expected to have $r > 1$). In this section, we shall therefore use the radius of gyration as our length scale and concentrate on the scaling behavior of fragments with $r < 1$.

Thus both the mass and growth probability measure can be characterized in terms of conservative stochastic multiplicative processes with a multiplicity ratio m dependent on the specific model simulated. For example, simulations on d dimensional hypercubic lattices would give $m = 2d - 1$, while off lattice two dimensional DLA implies $m = 4$ (any particle can be connected to a maximum of five other particles, one of which must be its parent). We shall define and study here two dimensional binary DLA.

Binary DLA is defined as normal DLA with the additional constraint that each node can have at most two subbranches—if a random walk reaches first a site neighboring a particle which already has two subbranches, it cannot stick there but must continue its walk until it reaches a site neighboring a particle with fewer descendants. As a consequence, binary DLA has a perfect binary tree topology with $m = 2$. Recursive algorithms can be used to analyze its SPD for both mass and growth measure and the fragments at each generation identified. In Fig. 1 we have plotted both a typical binary DLA cluster and its associated fragments at different generations. The cluster itself is shown in Fig. 1(a); in Fig. 1(b) circles of radius of gyration covering the fragments with $r < 1$ are plotted. The early generation fragments are coded a lighter shade of grey than the later ones for purposes of visualization. Binary DLA is a fractal in the same universality class as normal DLA (in fact its fractal dimension appears to be even closer to $D = \frac{5}{3}$ than normal DLA), but its multifractal properties can be analyzed more simply because of its binary tree topology.

One of the basic assumptions of this paper is that many physical stochastic multifractals are Markovian in construction and therefore have an SPD that is independent of generation. In Fig. 2 we plot the function $\ln P_{\text{mass}}(f, r)$ for clusters of 5000 and 10 000 particles using the mass measure. The abscissa corresponds to the length scale ratio r and the ordinate corresponds to the measure fraction f. The darker the grey level, the larger the magnitude of $P_{\text{mass}}(f, r)$ on a logarithmic scale. The white circles correspond to the curve $f = r^{5/3}$. An examination of all our data shows that while 1000 particle cluster data are fairly noisy, the data for larger clusters such as the 5000 and 10 000 particle data shown in Fig. 2 settle down to a universal form independent of cluster size and consistent with the assumption of a Markov process. This SPD, while peaked approximately around $f = r^{5/3}$ and diverging in magnitude near ($r = 0$, $f = 0$ and $r = 1$, $f = 1$), consistent with the dynamics being a H process with corner singularities of the type described in Sec. IX, nevertheless shows clear deviations from the limiting form. These deviations are describable in terms of B distributions: (i) a band of small but definitely nonzero probability exists at small r for all f; in other words, the dynamics of DLA fragmentation not only creates self-similar branches of various sizes but also a distribution of strong singularity "hot spots"—compact clusters containing much greater measure than appropriate for a homogeneous fractal with $D = \frac{5}{3}$; (ii) a weaker but still clear band of fragments with large r but small f can be seen; these are weak singularity "cold spots," very open

(a)



(b)

FIG. 1. (a) A typical binary DLA cluster. Like the normal DLA it is a fractal with $D \approx \frac{5}{3}$, but in contrast to normal DLA it has an exact binary tree structure. (b) Circles of radius of gyration $r < 1$ covering the mass measure fragments at each generation. The later generations are coded a darker gray for purposes of visualization.

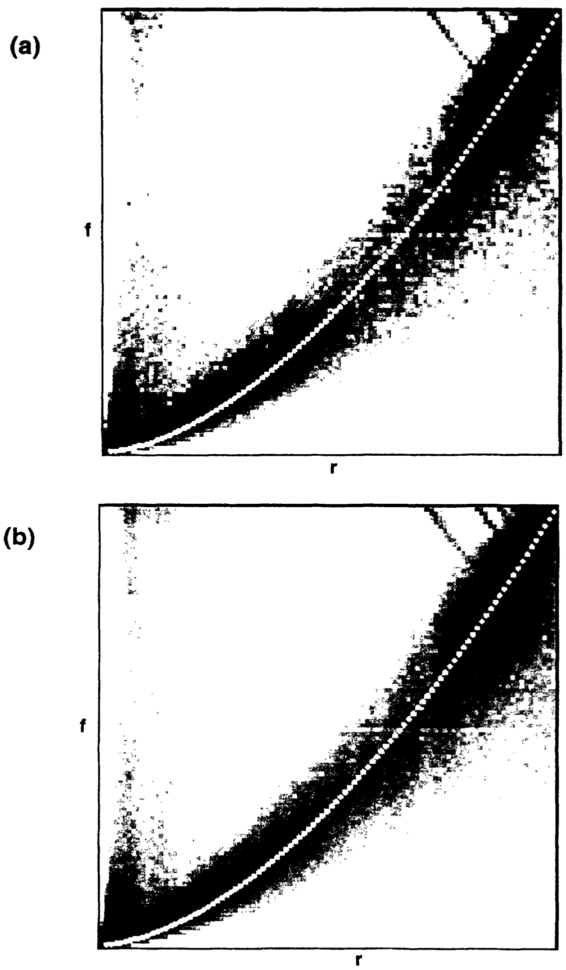


FIG. 2. $\ln P_{\text{mass}}(f,r)$ for clusters of (a) 5000 particles and (b) 10 000 particles, showing the Markovian dynamics of DLA. The abscissa corresponds to the length scale ratio r while the ordinate corresponds to the measure fraction f. The gray level is linear in $\ln P(f,r)$. The white circles correspond to the curve $f=r^{5/3}$.

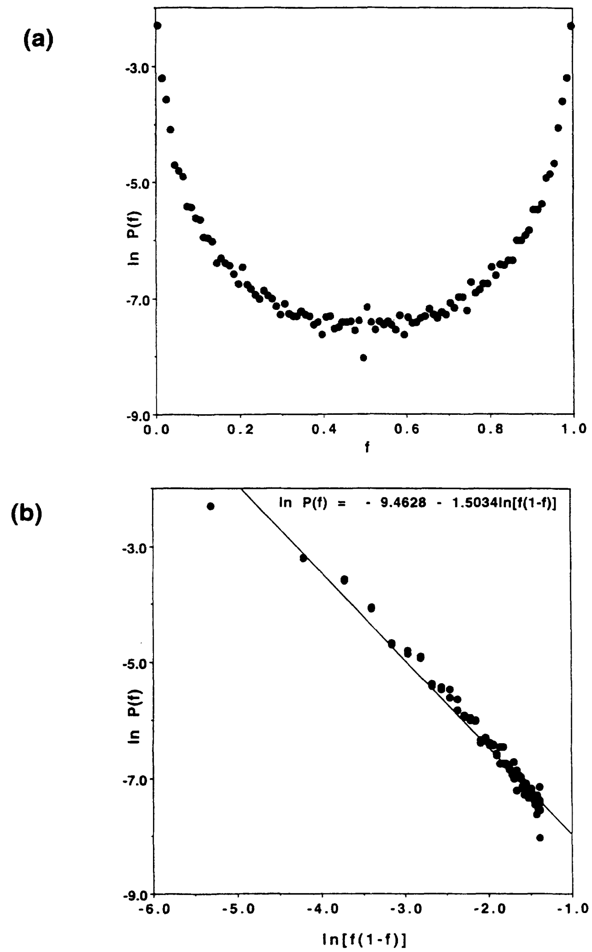


FIG. 3. The reduced SPD $P_{\text{mass}}(f)$ showing the symmetry about $f=\frac{1}{2}$. (a) A plot of $\ln P_{\text{mass}}(f)$ versus f; (b) a plot of $\ln P_{\text{mass}}(f)$ versus $\ln f(1-f)$. The straight line fit suggests that $P_{\text{mass}}(f) \approx A[f(1-f)]^{3/2}$ reasonably parametrizes the mass fraction distribution except at the limits $f \to 0$ and $f \to 1$.

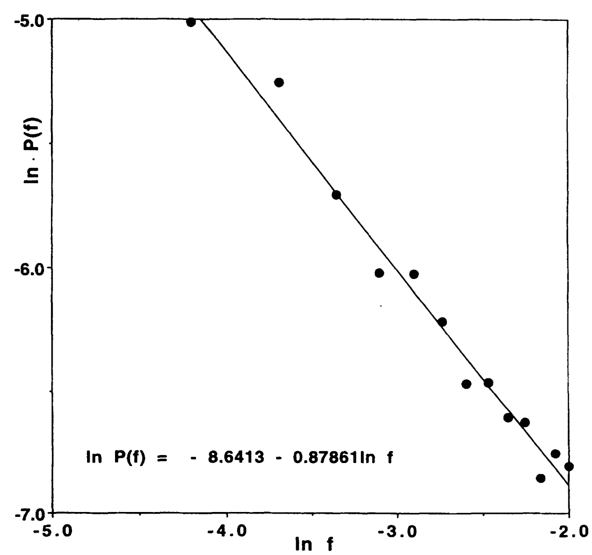


FIG. 4. The asymptotic scaling of $P_{\text{mass}}(f)$ for the mass measure as $f \to 0$. A least-squares fit yields $\mu \approx -0.88$.

dendritic branches of large relative radius but containing only a few particles; (iii) close to the region $r=1$, $f=1$ a tendril-like structure is appearing in the SPD which appears not be an artifact of the data. Its origin at present is not known.

These are all qualitative results for the mass measure SPD. To study mass measure scaling in greater detail we plot the probability $P_{\text{mass}}(f)$ of splitting into a mass fraction f in Fig. 3(a). In line with Eq. (88) this should be a symmetric function about $f=\frac{1}{2}$; indeed a reasonable fit to the complete curve [see Fig. 3(b)] is given by the symmetric function $P_{\text{mass}}(f) \approx A[f(1-f)]^{-1.5}$, though this form must break down close to $f \to 0$ and $f \to 1$ from the constraint of normalization. To study this asymptotic region more closely we have plotted $\ln P_{\text{mass}}(f)$ versus $\ln f$ as $f \to 0$ in Fig. 4. A least-squares fit to the data yields $\ln P(f) \approx -8.64 - 0.88 \ln f$. Though our data are not really good enough to be certain on this point, it does appear that for the mass measure $\mu \approx -0.88$ and certainly

$\mu > -1$. Thus the dynamics favors the creation of large and small mass fragments at the expense of similar sized fragments, but the central limit theorem is still obeyed by the mass measure of two dimensional binary DLA. Consequently no phase transitions will be observed in the mass multifractal spectrum for quenched exponents [see Eq. (70)]. For annealed exponents a phase transition should exist at $q_{\text{bottom}} = -(\mu+1) \approx -0.12$, but this may be very difficult to observe in numerical simulations because of the poor statistics for annealed averaging.

The mass measure is not the only measure that can be used to study DLA. Even more closely related to the dynamics is the growth measure associated with any given branch, which can be found by studying where random walks stick on the cluster. From this measure the growth SPD can be calculated and its logarithm $\ln P_{\text{growth}}(f,r)$ is plotted in Fig. 5 for both 5000 and 10 000 particle clusters with the grey level increasing with probability on a logarithmic scale. The Markovian nature of the process can be seen from the approximate invariance of the SPD with cluster size (there are still finite size effects observable, but it does appear to be settling down to a limiting distribution). The most obvious difference between Figs. 5 and 2 is that, whereas the mass SPD $\ln P_{\text{mass}}(f,r)$ appears to be approximately homogeneous with $f \approx r^{5/3}$, the maximum value of $\ln P_{\text{growth}}(f,r)$ as a function of r is approximately shaped like a reflected Z with divergences as $P_{\text{growth}}(f \to 0, r < \frac{1}{2})$ and $P_{\text{growth}}(f \to 1, r > \frac{1}{2})$. Therefore one branch will always tend to dominate the growth process. This result together with the singularity in the mass SPD at $r=0$, $f=0$ and $r=1$, $f=1$ suggests a "winner take all" dynamical mechanism, supporting the ideas of Halsey and Leibig [61] of competitive branch growth in DLA.

To study the growth SPD more quantitatively we have plotted in Fig. 6(a) $\ln P_{\text{growth}}(f)$. As expected this is a symmetric function about $f = \frac{1}{2}$, which can be fitted very well by the form $P(f) \approx A[f(1-f)]^{-1}$ over the entire range of growth fractions. This form does indeed suggest

(a)

(b)

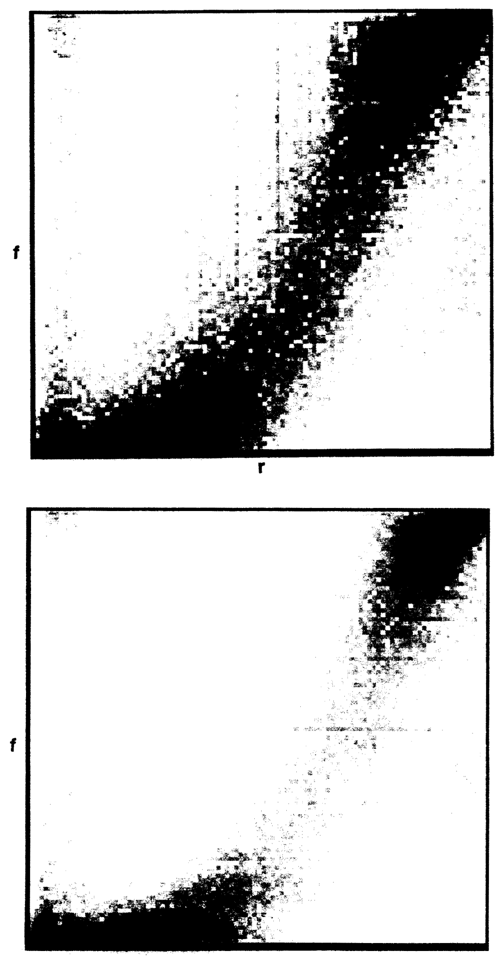


FIG. 5. $\ln P_{\text{growth}}(f,r)$ for clusters of (a) 5000 particles and (b) 10 000 particles, showing the Markovian dynamics of DLA. The abscissa corresponds to the length scale ratio r while the ordinate corresponds to the measure fraction f. The gray level is linear in $\ln P_{\text{growth}}(f,r)$.

(a)

(b)

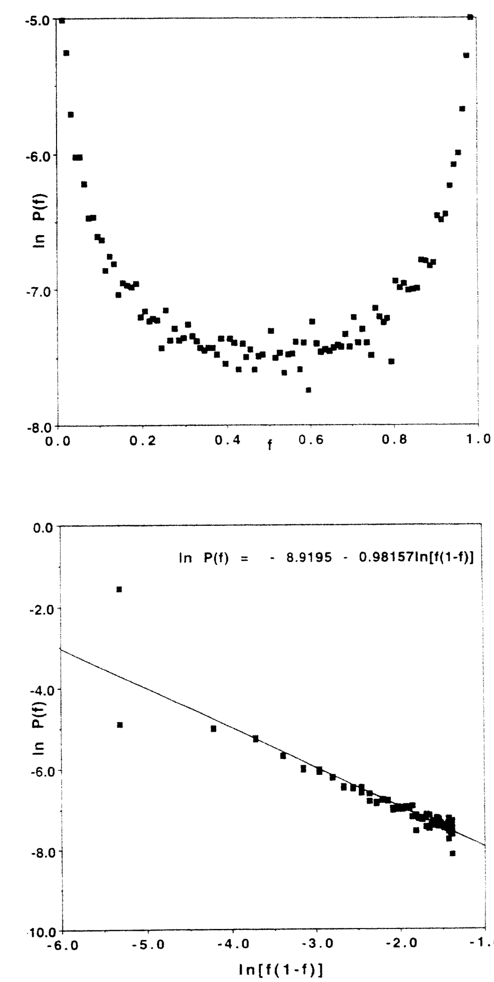


FIG. 6. The logarithm of the reduced SPD $\ln P_{\text{growth}}(f)$ plotted against (a) f and (b) $\ln[f(1-f)]$, showing the breakdown in the central limit theorem for growth. $P_{\text{growth}}(f)$ is a symmetric function about $f = \frac{1}{2}$, which can be fitted by the form $P(f) \approx A[f(1-f)]^{-1}$ over the entire range of f. Multifractal phase transitions will be observed at $q_{\text{bottom}}(n) = 0$ for all replica averages.

that for the growth measure $\mu=-1$ and the central limit theorem breaks down. Therefore multifractal phase transitions will be observed at $q_{\rm bottom}(n)=0$ for all replica averages [see Eq. (74)] of the growth probability including quenched exponents $q_{{\rm bottom},Q}=0$ [see Eq. (75)] in line with the numerical simulations of Schwarzer *et al.* [54] and Trunfio and Alstrøm [53] for quenched exponents. These results are also in agreement with Mandelbrot and Evertz [62,63], who have suggested $q_{{\rm bottom},Q}=0$. The results of Lee and Stanley [51], who found $q_{{\rm bottom},A}\approx-1.0$ for annealed exponents would imply $\mu\approx 0$ and this result would in turn imply $q_{{\rm bottom},Q}\approx-\infty$. Which set of results is correct depends on a delicate interpretation of the PSD for very low growth fractions [see Fig. 6(b)]: does the SPD continue to diverge as $P(f)\sim 1/f$ apart from logarithmic corrections as $f\to 0$ or does it cross over to some less singular behavior, specifically does $P_{\rm growth}(f=0)$ exist? Note that simulations of the minimal growth probability in DLA by Schwarzer *et al.*

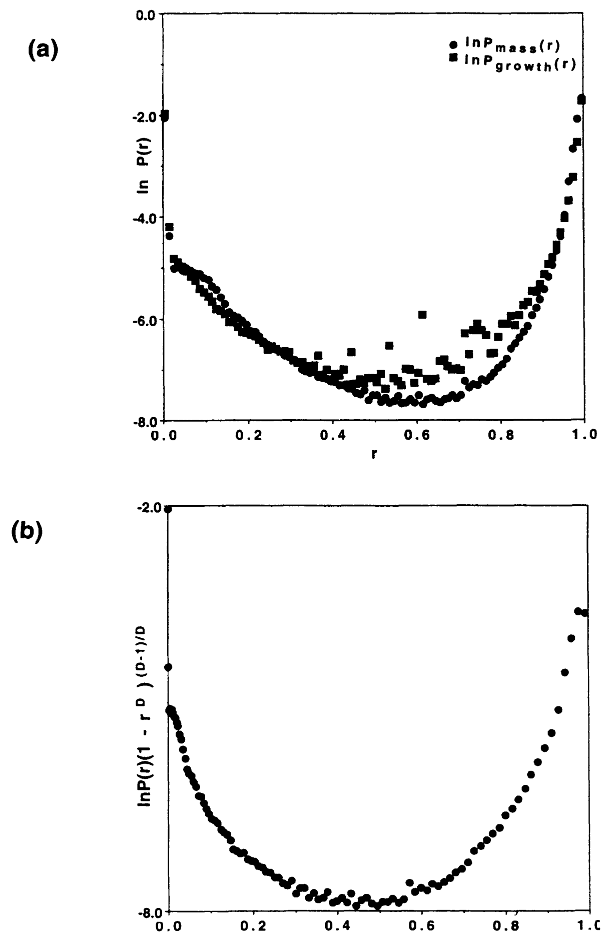


FIG. 7. The reduced SPD $P(r)$ showing the asymmetry about $r=\frac{1}{2}$, but suggestive of a scaling H process. (a) $\ln P(r)$ plotted against length scale r for both the growth and mass measures [$P(r)$ should be independent of the measure chosen]. (b) Replotted for the mass measure as $\ln[P_{\rm mass}(r)(1-r^D)^{(D-1)/D}]$ against r^D, the reflection symmetry about $r^D=\frac{1}{2}$ substantially increases.

[54] find a log normal (in the number of particles) decay in the minimal probability of the form given by Eq. (81), which suggests that the growth probabilities obey the central limit theorem and therefore $\mu>-1$. It is of course possible that normal and binary DLA have different singularities, but all previous analyses would suggest that any dependence on branching m is very weak and, in addition, normal DLA tends asymptotically to have an effective binary tree topology [61]. Therefore we think that more accurate numerical studies of binary DLA should settle this question.

The function $P(r)$ from its definition should be independent of the nature of the measure considered. Therefore in Fig. 7(a) we have overlayed the distribution $P(r)$ found for the growth measure over that for the mass measure. The agreement is reasonable and can be used to judge the noise fluctuations in our data. The dependence of $P(r)$ on length scale ratio r is not symmetric about $r=\frac{1}{2}$, but when replotted in Fig. 7(b) as suggested in Eq. (91) for scaling H processes—plot $P(r)(1-r^D)^{(D-1)/D}$ against r^D—the reflection symmetry about $r^D=\frac{1}{2}$ substantially increases. This symmetry is not exact because, as pointed out above, "hot" and "cold" fragments break this homogeneity. Nevertheless the approximate agreement observed confirms that DLA can be described to a good approximation as an H process.

The behavior of $P(r)$ for small r can be used to examine the question about whether an upper limit $q_{\rm top}(n)$ to the multifractal spectrum exists. The data for $P(r)$ from both the mass [$P_{\rm mass}(r)$] and growth [$P_{\rm growth}(r)$] simulations are plotted in Fig. 8. Is there a breakdown in the central limit theorem as far as the fragments ratios in DLA are concerned? The best straight line fit through all

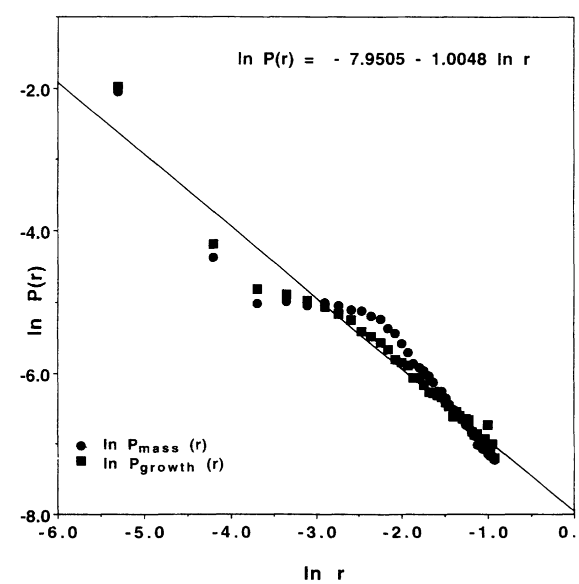


FIG. 8. $\ln P(r)$ plotted against length scale $\ln r$ for both the mass and growth measure data showing a breakdown of the central limit theorem for length scale ratio distributions in DLA. The best straight line fit $P(r)\sim r^{-1}$ suggests the existence of an upper phase transition at $q_{\rm top}(n)=1$.

the data would suggest $\nu=-1$, and this would imply that a phase transition should be observed in the multifractal spectrum at $q_{\text{top}}(n)=1$ [see Eq. (74)]. But this conclusion is also very delicate and it is also possible to see a crossover to a less singular limit in Fig. 8 provided the very smallest fragments are neglected. In this case $q_{\text{top}}(n)=\infty$.

XI. DISCUSSION

In this paper we have examined the characterization of stochastic multifractals by scaling probability distributions describing their creation by hierarchical Markovian random multiplicative processes. We have found that such a characterization is not only rich enough to encompass such multifractals as DLA, but it is much more informative about both structure and dynamics than any multifractal spectral representation whose simple behavior, constrained by the inequality relationships of Sec. V, washes out most interesting dynamical information except near phase transitions.

We have tried to emphasize in this paper universal classes of multiplicative stochastic fragmentation processes based on general properties of the SPD, especially its global singularities, independence or strong correlations in the SPD, and whether the process has conservation laws associated with it such as conservation of measure or volume.

That singularities appear in the SPD for real physical processes is apparent from both the mass and growth measures of DLA. Strong relationships exist between such SPD exponents and the averaging used to calculate generalized dimensions, on the one hand, and the dependence of both the positions $q_{\text{bottom}}(n),q_{\text{top}}(n)$ and the singularities in the $D(q,n)$ at such multifractal phase transitions. These relationships can be used to distinguish lognormal distributions (such as may be expected for the mass fractions in DLA) from those which disobey the central limit theorem (such as the growth measure fractions). Thus, though it is unlikely that the SPD can be recovered by inverting multifractal spectral data and the SPD is best measured directly as we have done here for DLA, it does appear that robust properties of the multifractal spectrum can be used to extract universal characteristics of the SPD describing the fragmentation dynamics.

The influence of the averaging used is crucial in the study of stochastic multifractals. The averaging process has a significant but well ordered effect on spectra as described by the inequalities derived in Sec. V. For conservative processes three qualitatively different types ordering appear: $D(q,n)\geq D(q,n')$ if $n>n'$ and $q<1$; $D(1,n)=D(1,n')$ if $n>n'$ and $q=1$; and $D(q,n)\leq D(q,n')$ if $n>n'$ and $q>1$. In addition, the positions and singularities of multifractal phase transitions are strongly, but in a universal manner, influenced by the nature of averaging.

Universal limits seem to exist for the generalized dimensions $D(q,n)$ asymptotically as $q\to\infty$ and $n\to\infty$. Specifically whether the generalized dimension remains finite or the way $D(q,n)\to 0$ depends on the behavior of the SPD at the edges of its domain of integration, while the manner in which it approaches zero depends on the averaging.

The influence of conservation laws is also significant. Measure conservation $\sum_\alpha f_\alpha=1$ such as that which occurs in DLA and in the inertial regime of turbulence is especially strong for the information dimension D_I, which uniquely is independent of replica averaging for conservative processes and does not exist for nonconservative ones. For space filling process $\sum_\alpha r_\alpha^d=1$, both conservative and nonconservative processes appear to have a weaker dependence on the averaging process involved than for processes with a fractal support.

The influence of the multiplicity of splittings m at each fragmentation generation is weak on the resulting multifractal distribution. The dependence is often logarithmic, and for a large class of models the position of the multifractal phase transitions is independent of m.

We also note that the usual α versus $f(\alpha)$ and D_q versus q formalisms are inadequate to describe stochastic multifractals. The spectrum depends in reality on the averaging process $D(q)\to D(q,n)$, while the idea of intertwined sets of measure singularities leading to the $f(\alpha)$ versus α description must be generalized to an ensemble of multifractals, each with its own distribution of singularities [an additional exponent $g(\mathbf{c})$ is therefore required, describing the probability that a member of the ensemble in question is described by a set of intensive order parameters $\mathbf{c}$ describing the stochasticity in the fragmentation process].

The differences between fragmentation processes where only finite-sized length scale ratios or measure fractions can occur (A processes) and those where singular distributions for very small or large fragments (B processes) exist are profound. A processes have a complete multifractal spectrum with no phase transition whatever the shape of the SPD. This situation changes drastically for type B processes for which the singularities in the measure result in multifractal phase transitions. For example, if $P(f,r)\sim f^{\mu_0}$ as $f\to 0$, then we may expect a multifractal phase transition at $q_{\text{bottom}}(n)=-(\mu_0+1)/n$, and this phase transition depends strongly on the averaging process.

There also exist strong correlations between the minimal fragment sizes (minimal growth probabilities in DLA, weakest vortices in turbulent flows) and the singularities of the SPD. The behavior of minimal size fragments is also closely connected to the existence of multifractal phase transitions. The minimal measure typically scales as a power law in the number of fragments in the case of complete spectra and log normally in the number of fragments for distributions obeying the central limit theorem, but has a stretched exponential form for distributions where breakdown occurs.

It would be of great interest to apply these results further both to the study of DLA and especially to inertial turbulence, where a direct extraction of $P(f,r)$ for the dissipation field encodes the manner in which energy is transported from large to small scales.

ACKNOWLEDGMENT

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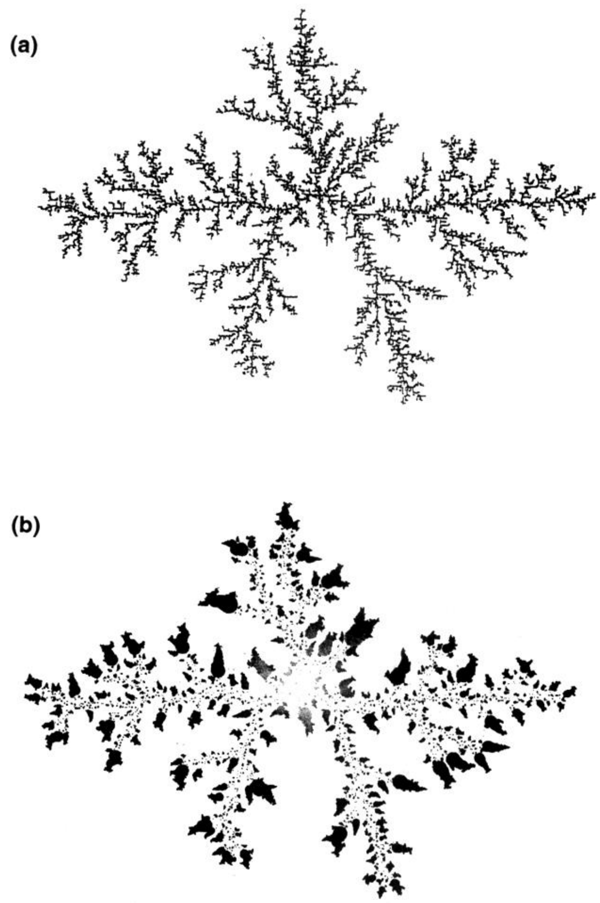

FIG. 1. (a) A typical binary DLA cluster. Like the normal DLA it is a fractal with $D \approx \frac{5}{3}$, but in contrast to normal DLA it has an exact binary tree structure. (b) Circles of radius of gyration $r < 1$ covering the mass measure fragments at each generation. The later generations are coded a darker gray for purposes of visualization.

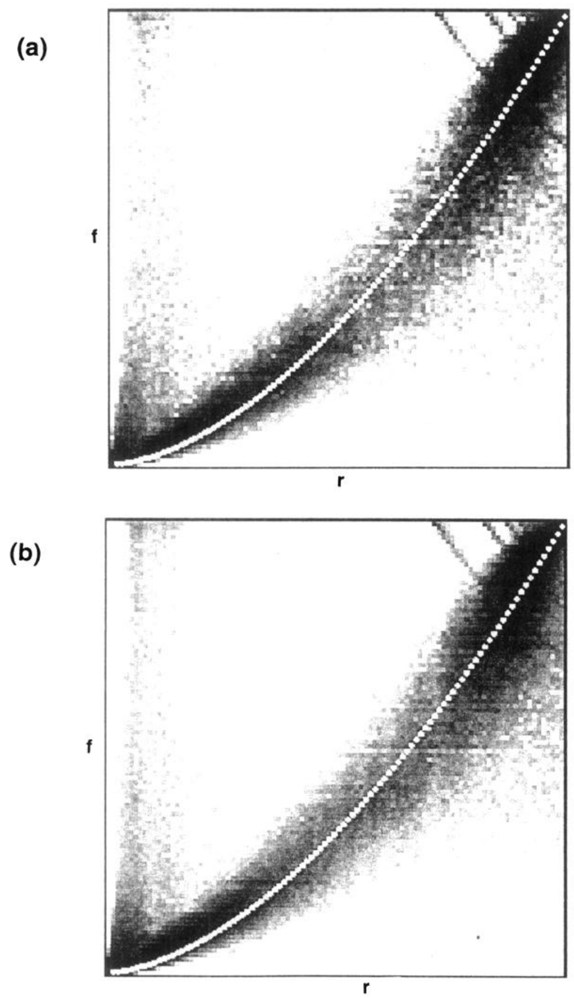

FIG. 2. $\ln P_{\text{mass}}(f,r)$ for clusters of (a) 5000 particles and (b) 10 000 particles, showing the Markovian dynamics of DLA. The abscissa corresponds to the length scale ratio r while the ordinate corresponds to the measure fraction f. The gray level is linear in $\ln P(f,r)$. The white circles correspond to the curve $f = r^{5/3}$.

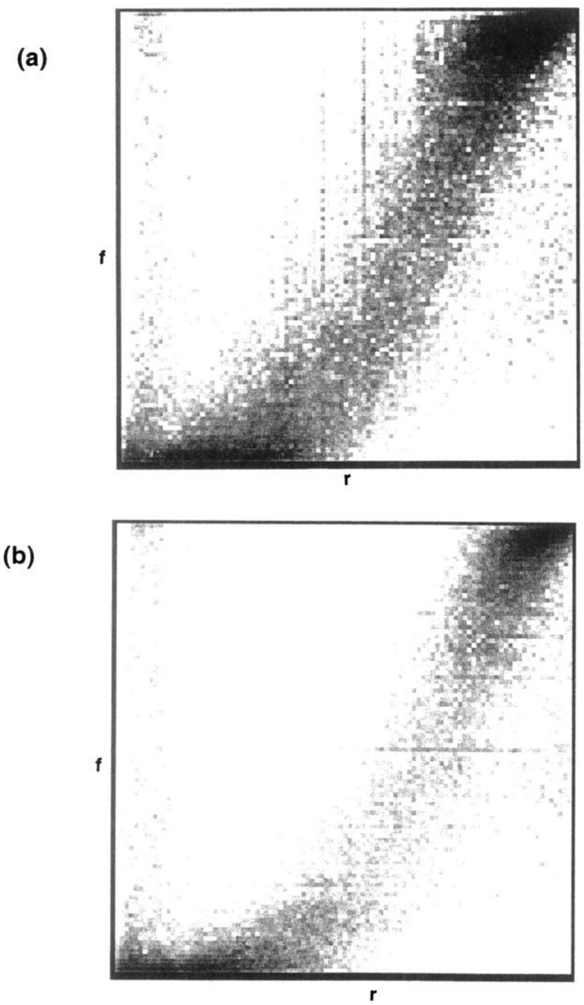

FIG. 5. $\ln P_{\text{growth}}(f,r)$ for clusters of (a) 5000 particles and (b) 10 000 particles, showing the Markovian dynamics of DLA. The abscissa corresponds to the length scale ratio r while the ordinate corresponds to the measure fraction f. The gray level is linear in $\ln P_{\text{growth}}(f,r)$.